

Higher-Order Linearisability

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Abstract. Linearisability is a central notion for verifying concurrent libraries: a given library is proven safe if its operational history can be rearranged into a new sequential one which, in addition, satisfies a given specification. Linearisability has been examined for libraries in which method arguments and method results are of ground type, including libraries parameterised with such methods. In this paper we extend linearisability to the general higher-order setting: methods can be passed as arguments and returned as values. A library may also depend on abstract methods of any order. We use this generalised notion to show correctness of several higher-order example libraries.

1 Introduction

Computer programs often take advantage of *libraries*, which are collections of routines, often of specialised nature, implemented to facilitate software development and, among others, code reuse and modularity. To support the latter, libraries should follow their specifications, which describe the range of expected behaviours the library should conform to for safe and correct deployment. Adherence to given specifications can be formalised using the classic notion of contextual approximation (refinement), which scrutinises the behaviour of code in any possible context. Unfortunately, the quantification makes it difficult to prove contextual approximations directly, which motivates research into sound techniques for establishing it.

In the concurrent setting, a notion that has been particularly influential is that of *linearisability* [12]. Linearisability requires that, for each history generated by a library, one should be able to find another history from the specification (its *linearisation*), which matches the former up to certain rearrangements of events. In the original formulation by Herlihy and Wang [12], these permutations were not allowed to disturb the order between library returns and client calls. Moreover, linearisations were required to be *sequential* traces, that is, sequences of method calls immediately followed by their returns. This notion of linearisability only applies to closed, i.e. fully implemented, libraries in which both method arguments and results are of *ground* types. The closedness limitation was lifted by Cerone, Gotsman and Yang [3], who extended the techniques to *parametric* libraries, whereby methods were divided into available routines (public methods) and unimplemented ones (abstract methods). However, both public and abstract methods were still restricted to first-order functions of type $\text{int} \rightarrow \text{int}$. In this paper, we make a further step forward and present linearisability for general higher-order concurrent libraries, where methods can be of arbitrary higher-order types. In doing so, we also propose a corresponding notion of sequential history for higher-order histories.

We examine libraries L that can interact with their environments by means of public and abstract methods: a library L with abstract methods of types $\Theta = \theta_1, \dots, \theta_n$ and

public methods $\Theta' = \theta'_1, \dots, \theta'_n$, is written as $L : \Theta \rightarrow \Theta'$. We shall work with arbitrary higher-order types generated from the ground types `unit` and `int`. Types in Θ, Θ' must always be function types, i.e. their order is at least 1.

A library L may be used in computations by placing it in a context that will keep on calling its public methods (via a client K) as well as providing implementations for the abstract ones (via a parameter library L'). The setting is depicted in Figure 1. Note that, as the library L interacts with K and L' , they exchange functions between each other. Consequently, in addition to K making calls to public methods of L and L making calls to its abstract methods, K and L' may also issue calls to functions that were passed to them as arguments during higher-order interactions. Analogously, L may call functions that were communicated to it via library calls.

Our framework is operational in flavour and draws upon concurrent [15,7] and operational game semantics [13,16,8]. We shall model library use as a game between two participants: *Player* (P), corresponding to the library L , and *Opponent* (O), representing the environment (L', K) in which the library was deployed. Each call will be of the form $\text{call } m(v)$ with the corresponding return of the shape $\text{ret } m(v)$, where v is a value. As we work in a higher-order framework, v may contain functions, which can participate in subsequent calls and returns. Histories will be sequences of *moves*, which are calls and returns paired with thread identifiers. A history is sequential just if every move produced by O is immediately followed by a move by P in the same thread. In other words, the library immediately responds to each call or return delivered by the environment. In contrast to classical linearisability, the move by O and its response by P need not be a call/return pair, as the higher-order setting provides more possibilities (in particular, the P response may well be a call). Accordingly, linearisable higher-order histories can be seen as sequences of atomic segments (linearisation points), starting at environment moves and ending with corresponding library moves.

In the spirit of [3], we are going to consider two scenarios: one in which K and L' share an explicit communication channel (the general case) as well as a situation in which they can only communicate through the library (the encapsulated case). Further, in the encapsulated case, we will handle the case in which extra closure assumptions can be made about the parameter library (the relational case). The restrictions can deal with a variety of assumptions on the use of parameter libraries that may arise in practice.

In each of the three cases, we shall present a candidate definition of linearisability and illustrate it with tailored examples. The suitability of each kind of linearisability will be demonstrated by showing that it implies the relevant form of contextual approximation (refinement). We shall also examine compositionality of the proposed concepts. One of our examples will discuss the correctness of an implementation of the flat-combining approach [11,3], adapted to higher-order types.

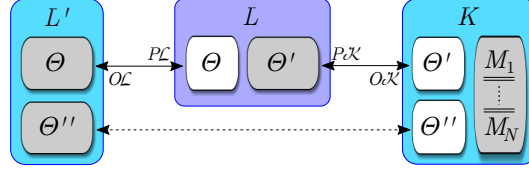


Fig. 1. A library $L : \Theta \rightarrow \Theta'$ in environment comprising a parameter library $L' : \emptyset \rightarrow \Theta, \Theta''$ and a client K of the form $\Theta', \Theta'' \vdash M_1 \parallel \dots \parallel M_N$.

2 Higher-order linearisability

As mentioned above, we examine libraries interacting with their context by means of abstract and public methods. In particular, we consider higher-order types given by the grammar on the left below, and let Meths be a set of *method names*.

$$\theta ::= \text{unit} \mid \text{int} \mid \theta \times \theta \mid \theta \rightarrow \theta \quad \text{Meths} = \bigsqcup_{\theta, \theta'} \text{Meths}_{\theta, \theta'} \quad v ::= () \mid i \mid m \mid (v, v)$$

Methods are ranged over by m (and variants), and each set $\text{Meths}_{\theta, \theta'}$ contains names for methods of type $\theta \rightarrow \theta'$. Finally, we let v range over computational *values*, which include a unit value, integers, methods, and pairs of values.

The framework of a higher-order library and its environment is depicted in Figure 1. Given $\Theta, \Theta' \subseteq \text{Meths}$, a library L is said to have type $\Theta \rightarrow \Theta'$ if it defines public methods with names (and types) as in Θ' , using abstract methods Θ . The environment of L consists of a *client* K (which invokes the public methods of Θ'), and a *parameter library* L' (which provides the code for the abstract methods Θ). In general, K and L' may be able to interact via a disjoint set of methods $\Theta'' \subseteq \text{Meths}$, to which L has no access.

In the rest of this paper we will be implicitly assuming that we work with a library L operating in an environment presented in Figure 1. The client K will consist of a fixed number N of concurrent threads. Next we introduce a notion of history tailored to the setting and define how histories can be linearised. In Section 3 we present the syntax for libraries and clients; and in Section 4 we define their semantics in terms of histories (and co-histories).

2.1 Higher-order histories

The operational semantics of libraries will be given in terms of *histories*, which are sequences of method calls and returns each decorated with a thread identifier and a *polarity index* XY , where $X \in \{O, P\}$ and $Y \in \{\mathcal{L}, \mathcal{K}\}$:

$$(t, \text{call } m(v))_{XY} \quad (t, \text{ret } m(v))_{XY}$$

We refer to decorated calls and returns like above as *moves*. Here, m is a method name and v is a value of matching type. The index XY is specifying which of the three entities (L, L', K) produces the move, and towards whom it is addressed.

- If $X = P$ then the move is issued by L . Moreover, if $Y = \mathcal{L}$ then it is addressed to L' ; otherwise, if $Y = \mathcal{K}$ then it is addressed to K .
- If $XY = O\mathcal{L}$ then the move is issued by L' , and is addressed to L .
- If $XY = O\mathcal{K}$ then the move is issued by K , and is addressed to L .

We can justify the choice of indices: the moves can be seen as defining a 2-player game between the library (L), which represents the *Proponent* player in the game, and its environment (L', K) that represents the *Opponent*. Moves played between L and L' are moreover decorated with \mathcal{L} ; whereas those between L and K have \mathcal{K} . Note that the possible interaction between L' and K is invisible to L and is therefore not accounted for in the game (but we will later see how it can affect it). We use O to refer to either OK or $O\mathcal{L}$, and P to refer to either PK or $P\mathcal{L}$.

Definition 1. [Prehistories] We define *prehistories* as sequences of moves derived by one of the following grammars,

$$\begin{aligned} H_{\text{pre}}^O &::= \epsilon \mid \text{call } m(v)_{OY} H_{\text{pre}}^P \text{ret } m(v')_{PY} H_{\text{pre}}^O \\ H_{\text{pre}}^P &::= \epsilon \mid \text{call } m(v)_{PY} H_{\text{pre}}^O \text{ret } m(v')_{OY} H_{\text{pre}}^P \end{aligned}$$

where, in each line, the two occurrences of $Y \in \{\mathcal{K}, \mathcal{L}\}$ and $m \in \text{Meths}$ must each match. Moreover, if $m \in \text{Meths}_{\theta, \theta'}$, the types of v, v' must match θ, θ' respectively.

The elements of H_{pre}^O are patterns of actions starting with an O -move, while those in H_{pre}^P start with a P -move. Note that, in each case, the polarities alternate and the polarities of calls and matching returns always match the pattern $(XY, X'Y)$ for $X \neq X'$.

Histories will be interleavings of prehistories tagged with thread identifiers (natural numbers) that satisfy a number of technical conditions. Given $h \in H_{\text{pre}}^{O/P}$ and $t \in \mathbb{N}$, we write $t \times h$ for h in which each call or return is decorated with t . We refer to such moves with $(t, \text{call } m(v))_{XY}$ or $(t, \text{ret } m(v))_{XY}$ respectively. If we only want to stress the X or Y membership, we shall drop Y or X respectively. Moreover, when no confusion arises, we may sometimes drop a move's polarity altogether.

Definition 2. [Histories] Given Θ, Θ' , the set of *histories* over $\Theta \rightarrow \Theta'$ is defined by:

$$\mathcal{H}_{\Theta, \Theta'} = \bigcup_{N \geq 0} \bigcup_{h_1, \dots, h_N \in H_{\text{pre}}^O} (1 \times h_1) \mid \dots \mid (N \times h_N)$$

where $(1 \times h_1) \mid \dots \mid (N \times h_N)$ is the set of all interleavings of $(1 \times h_1), \dots, (N \times h_N)$, satisfying the following conditions.

1. For any $s_1(t, \text{call } m(v))_{XY} s_2 \in \mathcal{H}_{\Theta, \Theta'}$:
 - either $m \in \Theta'$ and $XY = OK$, or $m \in \Theta$ and $XY = PL$,
 - or there is a move $(t', x')_{X'Y}$ in s_1 with $X \neq X'$, such that $x' \in \{\text{call } m'(v), \text{ret } m'(v)\}$ and v contains m .
2. For any $s_1(t, x)_{XY} s_2 \in \mathcal{H}_{\Theta, \Theta'}$, where $x \in \{\text{call } m(v), \text{ret } m(v)\}$ and v includes some $m' \in \text{Meths}$, m' must not occur in s_1 .

Condition 1 in the definition above requires that any call must refer to Θ or Θ' , or be introduced earlier as a higher-order argument or result. If the method is from Θ' , the call must be tagged with OK (i.e. issued by K). Dual constraints apply to Θ . If a method name does not come from Θ or Θ' , in order for the call $(t, \text{call } m(v))_{XY}$ to be valid, m must be introduced in an earlier action with the same tag Y but with the opposite tag X . Moreover, as specified by Condition 2, any action involving a higher-order value (i.e. a method name) in its argument or result must label it with a fresh name, one that has not been used earlier. This is done to enable the history to refer unambiguously to each method name encountered during the interaction.

We shall range over $\mathcal{H}_{\Theta, \Theta'}$ using h, s . The subscripts Θ, Θ' will often be omitted.

Remark 3. Histories will be used to define the semantics $\llbracket L \rrbracket$ of libraries (cf. Section 4). In particular, for each library $L : \Theta \rightarrow \Theta'$, we shall have $\llbracket L \rrbracket \subseteq \mathcal{H}_{\Theta, \Theta'}$.

```

1 public count, update;
2
3 Lock lock;
4 F := λx.0;
5
6 count = λi. (!F)i
7
8 update = λ(i, g). upd_r(i, g, (!F)i)
9
10 upd_r = λ(i, g, j).
11   let y = |gj| in
12   lock.acquire ();
13   let f = !F in
14   if (j==fi) then {
15     F := λx. if (x==i) then y else fx;
16     lock.release (); y }
17   else { lock.release (); upd_r(i, g, fi) }

```

Fig. 2. Multiset library L_{mset} . [$\text{count} : \text{int} \rightarrow \text{int}$, $\text{update} : \text{int} \times (\text{int} \rightarrow \text{int}) \rightarrow \text{int}$]

Example 4. Let $\Theta = \{m : \text{int} \rightarrow \text{int}\}$ and $\Theta' = \{m' : \text{int} \rightarrow \text{int}\}$. Note that single-threaded library histories from $\mathcal{H}_{\Theta, \Theta'}$ must have the shape $(1, \text{call } m'(i_1))_{OK} (1, \text{ret } m'(j_1))_{PK} \dots (1, \text{call } m'(i_k))_{OK} (1, \text{ret } m'(j_k))_{PK}$. In this case, the definition coincides with [3]. However, in general, our notion of histories is more liberal. For example, the sequence $(1, \text{call } m'(1))_{OK} (1, \text{call } m(2))_{PL} (1, \text{call } m'(3))_{OK} (1, \text{ret } m'(4))_{PK} (1, \text{ret } m(5))_{OL} (1, \text{ret } m'(6))_{PK}$ is in $\mathcal{H}_{\Theta, \Theta'}$, even though it is not allowed by Definition 1 of [3]. The sequence represents a scenario in which the public method m' is called for the second time before the first call is answered. In our higher-order setting, this scenario may arise if the parameter library communicates with the client and the communication includes a function that can issue a call to the public method m' .¹

Finally, we present a notion of sequential history, which generalises that of [12].

Definition 5. We call a history $h \in \mathcal{H}_{\Theta, \Theta'}$ *sequential* if it is of the form

$$h = (t_1, x_1)_{OY_1} (t_1, x'_1)_{PY'_1} \dots (t_k, x_k)_{OY_k} (t_k, x'_k)_{PY'_k}$$

for some $t_i, x_i, x'_i, Y_i, Y'_i$. We let $\mathcal{H}_{\Theta, \Theta'}^{\text{seq}}$ contain all sequential histories of $\mathcal{H}_{\Theta, \Theta'}$.

Next we consider our first example.

Example 6 (Multiset). Consider a concurrent multiset library L_{mset} that uses a private reference for storing the multiset's characteristic function. The implementation is given in Figure 2. This is a simplified version of the optimistic set algorithm of [10,19], albeit extended with a higher-order update method. The method computes the new value for element i without acquiring a lock on the characteristic function in the hope that when the lock is acquired the value at i will still be the same and the update can proceed (otherwise another attempt to update the value has to be made). The use of a single reference instead of a linked list means that memory safety is no longer problematic, so we focus on linearisability instead. Note we write $|j|$ for the absolute value of j .

Our verification goal will be to prove linearisability of L_{mset} to a specification $A_{\text{mset}} \subseteq \mathcal{H}_{\Theta, \Theta'}^{\text{seq}}$, where $\Theta = \{\text{count}, \text{update}\}$ (the method upd_r is private). A_{mset} certifies that L_{mset} correctly implements some integer multiset I whose elements change over time according to the moves in h . That is, for each history $h \in A_{\text{mset}}$ there is a multiset I that is empty at the start of h (i.e. $I(i) = 0$ for all i) and:²

¹ By comparison, in [3], each $(1, \text{call } m(v))_{PL}$ must be followed by some $(1, \text{ret } m(v'))_{OL}$.

² For a multiset I and a natural number i , we write $I(i)$ for the multiplicity of i in I ; moreover, we set $I[i \mapsto j]$ to be I with its multiplicity of i set to j .

- If I changes value between two consecutive moves in h then the second move is a P -move. In other words, the client cannot update the elements of I .
- Each call to *count* on argument i must be immediately followed by a return with value $I(i)$, and with I remaining unchanged.
- Each call to *update* on (i, m) must be followed by a call to m on i , with I unchanged. Then, m must later return with some value j . Assuming at that point the multiset will have value I' , if $I(i) = I'(i)$ then the next move is a return of the original *update* call, with value j ; otherwise, a new call to m on i is produced, and so on.

We formally define the specification next.

Let $\mathcal{H}_{\emptyset, \Theta}^{\circ}$ contain *extended histories* over $\emptyset \rightarrow \Theta$, which are histories where each move is accompanied by a multiset (i.e. the sequence consists of elements of the form $(t, x, I)_{XY}$). For each $s \in \mathcal{H}_{\emptyset, \Theta}^{\circ}$, we let $\pi_1(s)$ be the history extracted by projection, i.e. $\pi_1(s) \in \mathcal{H}_{\emptyset, \Theta}$. For each t , we let $s \upharpoonright t$ be the subsequence of s of elements with first component t . Writing \sqsubseteq for the prefix relation, and dropping the Y index from moves (Y is always \mathcal{K} here), we define $A_{\text{mset}}^{\circ} = \{\pi_1(s) \mid s \in A_{\text{mset}}^{\circ}\}$ where:

$$A_{\text{mset}}^{\circ} = \{s \in \mathcal{H}_{\emptyset, \Theta}^{\circ} \mid \pi_1(s) \in \mathcal{H}_{\emptyset, \Theta}^{\text{seq}} \wedge (\forall s'(-, I)_P(-, J)_O \sqsubseteq s. I = J) \wedge \forall t. s \upharpoonright t \in \mathcal{S}\}$$

and, for each t , the set of t -indexed histories \mathcal{S} is given by the following grammar.

$$\begin{aligned} \mathcal{S} &\rightarrow \epsilon \mid (t, \text{call } cnt(i), I)_O (t, \text{ret } cnt(I(i)), I)_P \mathcal{S} \\ &\quad \mid (t, \text{call } upd(i, m), I)_O \mathcal{M}_{I, J}^{i, j} (t, \text{ret } upd(|j|), J[i \mapsto |j|])_P \mathcal{S} \\ \mathcal{M}_{I, J}^{i, j} &\rightarrow (t, \text{call } m(I(i)), I)_P \mathcal{S} (t, \text{ret } m(j), J)_O \quad (\text{if } I(i) = J(i)) \\ \mathcal{M}_{I, J}^{i, j} &\rightarrow (t, \text{call } m(I(i)), I)_P \mathcal{S} (t, \text{ret } m(j'), J')_O \mathcal{M}_{J', J}^{i, j} \quad (\text{if } I(i) \neq J'(i)) \end{aligned}$$

By definition, the histories in A_{mset}° are all sequential. The elements of A_{mset}° carry along the multiset I that is being represented. The conditions on A_{mset}° stipulate that O cannot change the value of I , while the rest of the conditions above are imposed by the grammar for \mathcal{S} . With the notion of linearisability to be introduced next, it will be possible to show that $\llbracket L_{\text{mset}} \rrbracket$ indeed linearises to A_{mset} (see Section 5.1).

2.2 General linearisability

We begin with a definition of reorderings on histories.

Definition 7. Suppose $X, X' \in \{O, P\}$ and $X \neq X'$. Let $\triangleleft_{XX'} \subseteq \mathcal{H}_{\Theta, \Theta'} \times \mathcal{H}_{\Theta, \Theta'}$ be the smallest binary relation over $\mathcal{H}_{\Theta, \Theta'}$ satisfying

$$\begin{aligned} s_1(t', x')_{X'}(t, x)_{s_2} \triangleleft_{XX'} s_1(t, x)(t', x')_{X'} s_2 \\ s_1(t', x')(t, x)_X s_2 \triangleleft_{XX'} s_1(t, x)_X(t', x') s_2 \end{aligned}$$

where $t \neq t'$.

Intuitively, two histories h_1, h_2 are related by $\triangleleft_{XX'}$ if the latter can be obtained from the former by swapping two adjacent moves from different threads in such a way that, after the swap, an X -move will occur earlier or an X' -move will occur later. Note that, because of $X \neq X'$, the relation always applies to pairs of moves of the same polarity. On the other hand, we cannot have $s_1(t, x)_X(t', x')_{X'} s_2 \triangleleft_{XX'} s_1(t', x')_{X'}(t, x)_X s_2$.

Definition 8. [General Linearisability] Given $h_1, h_2 \in \mathcal{H}_{\Theta, \Theta'}$, we say that h_1 is *linearised* by h_2 , written $h_1 \sqsubseteq h_2$, if $h_1 \triangleleft_{PO}^* h_2$.

Given libraries $L, L' : \Theta \rightarrow \Theta'$ and set of sequential histories $A \subseteq \mathcal{H}_{\Theta, \Theta'}$ we write $L \sqsubseteq A$, and say that L *can be linearised to* A , if for any $h \in \llbracket L \rrbracket$ there exists $h' \in A$ such that $h \sqsubseteq h'$. Moreover, we write $L \sqsubseteq L'$ if $L \sqsubseteq \llbracket L' \rrbracket \cap \mathcal{H}_{\Theta, \Theta'}^{\text{seq}}$.

Remark 9. The definition above follows the classic definition from [12] and allows us to (first) express linearisability in terms of a given library L and a sequential specification A . The definition given in [3], on the other hand, expresses linearisability as a relation between two libraries. This is catered for at the end of Definition 8. Explicitly, we have that $L \sqsubseteq L'$ if for all $h \in \llbracket L \rrbracket$ there is some sequential $h' \in \llbracket L' \rrbracket$ such that $h \sqsubseteq h'$.

Example 10. Let $\Theta = \{m : \text{int} \rightarrow \text{int}\}$ and $\Theta' = \{m' : \text{int} \rightarrow \text{int}\}$. First, consider $h, h_1, h_2 \in \mathcal{H}_{\Theta, \Theta'}$ given by:

$$\begin{aligned} h &= (1, \text{call } m(1))_{OK} (1, \text{ret } m(2))_{PK} (2, \text{call } m(3))_{OK} (2, \text{ret } m(4))_{PK} \\ h_1 &= (1, \text{call } m(1))_{OK} (2, \text{call } m(3))_{OK} (1, \text{ret } m(2))_{PK} (2, \text{ret } m(4))_{PK} \\ h_2 &= (2, \text{call } m(3))_{OK} (1, \text{call } m(1))_{OK} (2, \text{ret } m(4))_{PK} (1, \text{ret } m(2))_{PK} \end{aligned}$$

The histories are related in the following ways: for any $i = 1, 2$, we have $h \triangleleft_{OP}^* h_i$ and $h_i \triangleleft_{PO}^* h$. Moreover, $h_1 \triangleleft_{XX'}^* h_2$ and $h_2 \triangleleft_{XX'}^* h_1$ for any $X \neq X'$. Note that we do *not* have $h \triangleleft_{PO}^* h_i$ or $h_i \triangleleft_{OP}^* h$.

Consider now $h_3, h_4 \in \mathcal{H}_{\Theta, \Theta'}$ given respectively by:

$$\begin{aligned} h_3 &= (1, \text{call } m(1))_{OK} (1, \text{call } m'(2))_{PL} (1, \text{ret } m'(3))_{OL} (1, \text{ret } m(4))_{PK} \\ &\quad (2, \text{call } m(5))_{OK} (2, \text{call } m'(6))_{PL} (2, \text{ret } m'(7))_{OL} (2, \text{ret } m(8))_{PK} \\ h_4 &= (1, \text{call } m(1))_{OK} (2, \text{call } m(5))_{OK} (1, \text{call } m'(2))_{PL} (1, \text{ret } m'(3))_{OL} \\ &\quad (2, \text{call } m'(6))_{PL} (2, \text{ret } m'(7))_{OL} (2, \text{ret } m(8))_{PK} (1, \text{ret } m(4))_{PK} \end{aligned}$$

Observe that $h_3 \triangleleft_{OP}^* h_4$ (and, thus, $h_4 \triangleleft_{PO}^* h_3$). However, we do not have $h_4 \triangleleft_{OP}^* h_3$ or $h_3 \triangleleft_{PO}^* h_4$.

Regarding linearisability, we can make the following remarks.

- Observe that in histories from $\mathcal{H}_{\Theta, \Theta'}$, we shall have the following actions: $\text{call } m'(i)_O$ and $\text{ret } m'(j)_P$. Thus, \triangleleft_{PO}^* cannot swap $(t, \text{ret } m'(j))$ with $(t', \text{call } m'(i))$, as in the standard definition of linearisability [12].
- When $\mathcal{H}_{\Theta, \Theta'}$ is considered, the available actions are $\text{call } m'(i)_O$, $\text{ret } m(j)_O$ and $\text{call } m(i)_P$, $\text{ret } m'(j)_P$. Then \triangleleft_{PO}^* coincides with Definition 2 of [3] for second-order libraries.

2.3 Encapsulated linearisability

A different notion of linearisability will be applicable in cases where the parameter library L' of Figure 1 is encapsulated, that is, the client K can have no direct access to it (i.e. $\Theta'' = \emptyset$). In particular, we shall impose an extra condition on histories in order to reflect the more restrictive nature of interaction. Specifically, in addition to sequentiality in every thread, we shall disallow switches between \mathcal{L} and \mathcal{K} components by O .

Definition 11. We call a history $h \in \mathcal{H}_{\Theta, \Theta'}$ **encapsulated** if, for each thread t , if $h = s_1(t, x)_{PY} s_2(t, x')_{OY'} s_3$ and moves from t are absent from s_2 then $Y = Y'$. Moreover, we set $\mathcal{H}_{\Theta, \Theta'}^{\text{enc}} = \{h \in \mathcal{H}_{\Theta, \Theta'} \mid h \text{ encapsulated}\}$ and $\llbracket L \rrbracket_{\text{enc}} = \llbracket L \rrbracket \cap \mathcal{H}_{\Theta, \Theta'}^{\text{enc}}$ (if $L : \Theta \rightarrow \Theta'$).

We define the corresponding linearisability notion as follows.

Definition 12. [Enc-linearisability] Let $\diamond \subseteq \mathcal{H}_{\Theta, \Theta'} \times \mathcal{H}_{\Theta, \Theta'}$ be the smallest binary relation on $\mathcal{H}_{\Theta, \Theta'}$ such that

$$s_1(t, m)_Y(t', m')_{Y'} s_2 \quad \diamond \quad s_1(t', m')_{Y'}(t, m)_Y s_2$$

for any $Y, Y' \in \{\mathcal{K}, \mathcal{L}\}$ such that $Y \neq Y'$ and $t \neq t'$.

Given $h_1, h_2 \in \mathcal{H}_{\Theta, \Theta'}^{\text{enc}}$, we say that h_1 is *enc-linearised* by h_2 , and write $h_1 \sqsubseteq_{\text{enc}} h_2$, if $h_1(\triangleleft_{PO} \cup \diamond)^* h_2$ and h_2 is sequential.

A library $L : \Theta \rightarrow \Theta'$ can be *enc-linearised to* A , written $L \sqsubseteq_{\text{enc}} A$, if $A \subseteq \mathcal{H}_{\Theta, \Theta'}^{\text{seq}} \cap \mathcal{H}_{\Theta, \Theta'}^{\text{enc}}$ and for any $h \in \llbracket L \rrbracket_{\text{enc}}$ there exists $h' \in A$ such that $h \sqsubseteq_{\text{enc}} h'$. Moreover, we write $L \sqsubseteq_{\text{enc}} L'$ if $L \sqsubseteq_{\text{enc}} \llbracket L' \rrbracket_{\text{enc}} \cap \mathcal{H}_{\Theta, \Theta'}^{\text{seq}}$.

Remark 13. Recall Θ, Θ' from Example 10. Note that histories may contain the following actions only: call $m'(i)_{OK}$, ret $m(i)_{OL}$, call $m(i)_{PL}$, ret $m'(i)_{PK}$. Then $(\triangleleft_{PO} \cup \diamond)^*$ preserves the order between call $m(i)_{PL}$ and ret $m(i)_{OL}$ as well as that between ret $m'(i)_{PK}$ and call $m'(i)_{OK}$, i.e. it coincides with Definition 3 of [3].

Example 14 (Parameterised multiset). We revisit the multiset library of Example 6 and extend it with an abstract method *foo* and a corresponding update method *update_enc* which performs updates using *foo* as the value-updating function. In contrast to the *update* method of L_{mset} , the method *update_enc* is not optimistic: it retrieves the lock upon its call, and only releases it before return. In particular, the method calls *foo* while it retains the lock. We call this library $L_{\text{mult2}} : \emptyset \rightarrow \Theta'$, with $\Theta' = \{\text{count}, \text{update}, \text{update_enc}\}$.

Observe that, were *foo* able to externally call *update*, we would reach a deadlock: *foo* would be keeping the lock while waiting for the return of a method that requires the lock. On the other hand, if the library is encapsulated then the latter scenario is not plausible. In such a case, L_{mult2} linearises to the specification A_{mult2} which is defined as follows (cf. Example 6). Let $A_{\text{mset2}} = \{\pi_1(s) \mid s \in A_{\text{mset2}}^\circ\}$ where:

$$A_{\text{mset2}}^\circ = \{s \in \mathcal{H}_{\emptyset, \Theta'}^\circ \mid \pi_1(s) \in \mathcal{H}_{\emptyset, \Theta'}^{\text{seq}} \wedge (\forall s'(-, I)_P(-, J)_O \sqsubseteq s. I = J) \wedge \forall t. s \upharpoonright t \in \mathcal{S}\}$$

and the set \mathcal{S} is now given by the grammar of Example 6 extended with the rule:

$$\mathcal{S} \rightarrow (t, \text{call } \text{upd_enc}(i), I)_{OK} (t, \text{call } \text{foo}(i), I)_{PL} (t, \text{ret } \text{foo}(j), I)_{OL} (t, \text{ret } \text{upd_enc}(|j|), I')_{PK} \mathcal{S}$$

with $I' = I[i \mapsto |j|]$. Linearisability is shown in Section 5.1.

2.4 Relational linearisability

Finally, we consider a special subcase of the encapsulated case, in which parameter libraries must satisfy additional constraints, specified through relational closure. This notion is desirable in cases when unconditional encapsulated linearisability does not hold, yet one may want to show that it would hold conditionally on the parameter library. The condition on the parameter library $L' : \emptyset \rightarrow \Theta$ is termed as a relation $\mathcal{R} \subseteq \mathcal{H}_{\emptyset, \Theta} \times \mathcal{H}_{\emptyset, \Theta}$.

<pre> 1 public count, update, update_enc; 2 abstract foo; 3 Lock lock; 4 F := λx.0; 5 6 count = λi. (!F)i 7 8 update = λ(i, g). upd_r(i, g, (!F)i) </pre>	<pre> 9 upd_r = λ(i, g, j). 10 ... 20 update_enc = λi. 21 lock.acquire (); 22 let y = foo ((!F)i) in 23 let f = !F in 24 F := λx. if (x==i) then y else fx; 25 lock.release (); y </pre>
---	--

Fig. 3. Parameterised multiset library L_{mset2} . [$\text{foo}, \text{update_enc} : \text{int} \rightarrow \text{int}$, lines 10-16 as in Fig. 2]

Definition 15. Given a history $h \in \mathcal{H}_{\Theta, \Theta'}$ and $X \in \{\mathcal{K}, \mathcal{L}\}$, we write $h \upharpoonright X$ for the subsequence of h with moves whose second index is X .

Given a sequence of moves s , we write \bar{s} for the sequence obtained from s by simply swapping the O/P indexes in its moves (e.g. $\overline{x_{OL} y_{PK} z_{PK}} = x_{PL} y_{OK} z_{OK}$).

Definition 16. [Relational linearisability] Let $\mathcal{R} \subseteq \mathcal{H}_{\Theta, \Theta} \times \mathcal{H}_{\Theta, \Theta}$ be a set closed under permutations of names in $\text{Meths} \setminus \Theta$. Given $h_1, h_2 \in \mathcal{H}_{\Theta, \Theta'}^{\text{enc}}$, we say that h_1 is \mathcal{R} -linearised by h_2 , written $h_1 \sqsubseteq_{\mathcal{R}} h_2$, if h_2 is sequential and $(h_1 \upharpoonright \mathcal{K}) \triangleleft_{PO}^* (h_2 \upharpoonright \mathcal{K})$ and $(\bar{h}_1 \upharpoonright \mathcal{L}) \mathcal{R} (\bar{h}_2 \upharpoonright \mathcal{L})$.

Given $L : \Theta \rightarrow \Theta'$ and $A \subseteq \mathcal{H}_{\Theta, \Theta}^{\text{seq}} \cap \mathcal{H}_{\Theta, \Theta}^{\text{enc}}$, we write $L \sqsubseteq_{\mathcal{R}} A$ if for any $h \in \llbracket L \rrbracket_{\text{enc}}$ there exists $h' \in A$ such that $h \sqsubseteq_{\mathcal{R}} h'$. Moreover, $L \sqsubseteq_{\mathcal{R}} L'$ if $L \sqsubseteq \llbracket L' \rrbracket_{\text{enc}} \cap \mathcal{H}_{\Theta, \Theta'}^{\text{seq}}$.

Note the above permutation-closure requirement: if $h \mathcal{R} h'$ and π is a (type-preserving) permutation on $\text{Meths} \setminus \Theta$ then $\pi(h) \mathcal{R} \pi(h')$. The requirement adheres to the fact that, apart from the method names from a library interface, the other method names in its semantics are arbitrary and can be freely permuted without any observable effect. Thus, \mathcal{R} in particular should not be distinguishing between such names.

Our third example extends the flat-combining case study from [3] by lifting it to higher-order types.

Example 17. Flat combining [11] is a synchronisation paradigm that advocates the use of single thread holding a global lock to process requests of all other threads. To facilitate this, threads share an array to which they write the details of their requests and wait either until they acquire a lock or their request has been processed by another thread. Once a thread acquires a lock, it executes all requests stored in the array and the outcomes are written to the shared array for access by the requesting threads.

The authors of [3] analysed a parameterised library that reacts to concurrent requests by calling the corresponding abstract methods subject to mutual exclusion. In Figure 4, we present the code adapted to arbitrary higher-order types, which is possible thanks to the presence of higher-order references in our framework. We assume:

$$\Theta = \{m_i \in \text{Meths}_{\theta_i, \theta'_i} \mid 1 \leq i \leq k\} \quad \Theta' = \{m'_i \in \text{Meths}_{\theta_i, \theta'_i} \mid 1 \leq i \leq k\}$$

Thus the setup of [3] corresponds to $\theta_i = \theta'_i = \text{int}$. The library L_{fc} is built following the flat combining approach and, on acquisition of a lock, the winning thread acts as a combiner of all registered requests. Note that the requests will be attended to one after

<pre> 1 abstract m_i; public m'_i; ...; 2 Lock $lock$; 3 struct { op, $parm$, $wait$, $retv$ } 4 $requests[N]$; 5 6 $m'_i = \lambda z.$ 7 $requests[t_{id}].op := i$; 8 $requests[t_{id}].parm := z$; 9 $requests[t_{id}].wait := 1$; </pre>	<pre> 10 while ($requests[t_{id}].wait$) 11 if ($lock.tryacquire()$) { 12 for ($t=0$; $t < N$; $t++$) 13 if ($requests[t].wait$) { 14 let $j = requests[t].op$ in 15 $requests[t].retv := m_j(requests[t].parm)$; 16 $requests[t].wait := 0$ 17 }; $lock.release()$ }; 18 $requests[t_{id}].retv$; </pre>
--	---

Fig. 4. Flat combination library L_{fc} .

another (thus guaranteeing mutual exclusion) and only one lock acquisition will suffice to process one array of requests.

In Section 5.2, we shall show that L_{fc} can be \mathcal{R} -linearised to the specification given by the sequential histories of the library L_{spec} that implements m'_i as follows:

```

6   $m'_i = \lambda z. ( lock.acquire(); \text{let } result = m_i(z) \text{ in } lock.release(); result )$ 

```

Thus, each abstract call in L_{spec} is protected by a lock.

3 Library syntax

We now look at the concrete syntax of libraries and clients. Libraries comprise collections of typed methods. The order³ of their argument and result types is unrestricted and will adhere to the grammar: $\theta ::= \text{unit} \mid \text{int} \mid \theta \rightarrow \theta \mid \theta \times \theta$.

We shall use three disjoint enumerable sets of names, referred to as Vars, Meths and Refs, to name respectively variables, methods and references. x, f (and their decorated variants) will be used to range over Vars; m will range over Meths and r over Refs.

Methods and references are implicitly typed, that is, we assume

$$\text{Meths} = \biguplus_{\theta, \theta'} \text{Meths}_{\theta, \theta'} \quad \text{Refs} = \text{Refs}_{\text{int}} \uplus \biguplus_{\theta, \theta'} \text{Refs}_{\theta, \theta'}$$

where $\text{Meths}_{\theta, \theta'}$ contains names for methods of type $\theta \rightarrow \theta'$, Refs_{int} contains names of integer references and $\text{Refs}_{\theta, \theta'}$ contains names for references to methods of type $\theta \rightarrow \theta'$. We write \uplus to stress the disjointness of sets in their union.

The syntax for building libraries is defined in Figure 5. Thus, each library L begins with a series of method declarations (public or abstract) followed by a block B consisting of method implementations ($m = \lambda x.M$) and reference initialisations ($r := i$ or $r := \lambda x.M$). Our typing rules will ensure that each public method must be implemented within the block, in contrast to abstract methods. On the other hand, a client K consists of a parallel composition of closed terms.

Terms M specify the shape of allowable method bodies. $()$ is the skip command, i ranges over integers, t_{id} is the current thread identifier and \oplus represents standard arithmetic operations. Thanks to higher-order references, we can simulate divergence

³ Type order is defined by $\text{ord}(\text{unit}) = \text{ord}(\text{int}) = 0$, $\text{ord}(\theta_1 \times \theta_2) = \max(\text{ord}(\theta_1), \text{ord}(\theta_2))$, $\text{ord}(\theta_1 \rightarrow \theta_2) = \max(\text{ord}(\theta_1) + 1, \text{ord}(\theta_2))$.

<i>Libraries</i>	$L ::= B \mid \text{abstract } m; L \mid \text{public } m; L$	<i>Clients</i>	$K ::= M \parallel \dots \parallel M$
<i>Blocks</i>	$B ::= \epsilon \mid m = \lambda x.M; B \mid r := \lambda x.M; B \mid r := i; B$		
<i>Terms</i>	$M ::= () \mid i \mid t_{\text{id}} \mid x \mid m \mid M \oplus M \mid \text{if } M \text{ then } M \text{ else } M \mid \langle M, M \rangle \mid \pi_1 M \mid \pi_2 M$ $\mid \lambda x^\theta.M \mid xM \mid mM \mid \text{let } x = M \text{ in } M \mid r := M \mid !r$		
<hr/>			
$\Gamma \vdash () : \text{unit}$	$\Gamma \vdash i : \text{int}$	$\Gamma \vdash t_{\text{id}} : \text{int}$	$\frac{\Gamma(x) = \theta}{\Gamma \vdash x : \theta} \quad \frac{m \in \text{Meths}_{\theta, \theta'}}{\Gamma \vdash m : \theta \rightarrow \theta'} \quad \frac{\Gamma \vdash M_1, M_2 : \text{int}}{\Gamma \vdash M_1 \oplus M_2 : \text{int}}$
$\frac{\Gamma \vdash M : \text{int} \quad \Gamma \vdash M_0, M_1 : \theta}{\Gamma \vdash \text{if } M \text{ then } M_1 \text{ else } M_0 : \theta}$	$\frac{\Gamma \vdash M_i : \theta_i \quad (i = 1, 2)}{\Gamma \vdash \langle M_1, M_2 \rangle : \theta_1 \times \theta_2}$	$\frac{\Gamma \vdash \langle M_1, M_2 \rangle : \theta_1 \times \theta_2}{\Gamma \vdash \pi_i M : \theta_i \quad (i = 1, 2)}$	$\frac{r \in \text{Refs}_{\text{int}}}{\Gamma \vdash !r : \text{int}}$
$\frac{r \in \text{Refs}_{\text{int}} \quad \Gamma \vdash M : \text{int}}{\Gamma \vdash r := M : \text{unit}}$	$\frac{r \in \text{Refs}_{\theta, \theta'}}{\Gamma \vdash !r : \theta \rightarrow \theta'}$	$\frac{r \in \text{Refs}_{\theta, \theta'} \quad \Gamma \vdash M : \theta \rightarrow \theta'}{\Gamma \vdash r := M : \text{unit}}$	$\frac{\Gamma, x : \theta \vdash M : \theta'}{\Gamma \vdash \lambda x^\theta.M : \theta \rightarrow \theta'}$
$\frac{\Gamma \vdash M : \theta \quad \Gamma, x : \theta \vdash N : \theta'}{\Gamma \vdash \text{let } x = M \text{ in } N : \theta'}$	$\frac{\Gamma(x) = \theta \rightarrow \theta' \quad \Gamma \vdash M : \theta}{\Gamma \vdash xM : \theta'}$	$\frac{m \in \text{Meths}_{\theta, \theta'} \quad \Gamma \vdash M : \theta}{\Gamma \vdash mM : \theta'}$	
<hr/>			
$\frac{\vdash_B \epsilon : \emptyset}{\vdash_B m = \lambda x.M; B : \Theta \uplus \{m\}}$	$\frac{m \in \text{Meths}_{\theta, \theta'} \quad x : \theta \vdash M : \theta' \quad \vdash_B B : \Theta}{\vdash_B m = \lambda x.M; B : \Theta \uplus \{m\}}$	$\frac{r \in \text{Refs}_{\theta, \theta'} \quad x : \theta \vdash M : \theta' \quad \vdash_B B : \Theta}{\vdash_B r := \lambda x.M; B : \Theta}$	
$\frac{r \in \text{Refs}_{\text{int}} \quad \vdash_B B : \Theta}{\vdash_B r := i; B : \Theta}$	$\frac{\vdash_B B : \Theta}{\text{Meths}(B) \vdash_L B : \emptyset \rightarrow \Theta}$	$\frac{\Theta \uplus \{m\} \vdash_L L : \Theta' \rightarrow \Theta'' \quad m \in \Theta''}{\Theta \vdash_L \text{public } m; L : \Theta' \rightarrow \Theta''}$	
$\frac{\Theta \uplus \{m\} \vdash_L L : \Theta' \rightarrow \Theta'' \quad m \notin \Theta''}{\Theta \vdash_L \text{abstract } m; L : \Theta' \uplus \{m\} \rightarrow \Theta''}$	$\frac{\text{Meths}(M_j) \subseteq \Theta \quad \vdash M_j : \text{unit} \quad (j = 1, \dots, N)}{\Theta \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}}$		

Fig. 5. Library syntax and typing rules for terms (\vdash), blocks (\vdash_B), libraries (\vdash_L), clients (\vdash_K).

by $(!r)()$, where $r \in \text{Refs}_{\text{unit}, \text{unit}}$ is initialised with $\lambda x^{\text{unit}}.(!r)()$. Similarly, after $r := \lambda x^{\text{unit}}.\text{let } y = M \text{ in } (\text{if } y \text{ then } (N; (!r)()) \text{ else } ())$, while $M N$ can be simulated by $(!r)()$. We shall also use the standard derived syntax for sequential composition, i.e. $M; N$ stands for $\text{let } x = N \text{ in } M$, where x does not occur in M .

Remark 18. In Section 2 we used lock-related operations in our example libraries (*acquire*, *tryacquire*, *release*), on the understanding that they can be coded using shared memory. Similarly, the array of Example 17 can be constructed using references.

For each term M , we write $\text{Meths}(M)$ for the set of method names occurring in M . We also use the same notation for method names in blocks and libraries. Terms are typed in environments $\Gamma = \{x_1 : \theta_1, \dots, x_n : \theta_n\}$ assigning types to their free variables.

Method blocks are typed through judgements $\vdash_B B : \Theta$, where $\Theta \subseteq \text{Meths}$. The judgements collect the names of methods defined in a block as well as making sure that the definitions respect types and are not duplicated. Also, any initialisation statements will be scrutinised for type compliance.

Finally, we type libraries using statements of the form $\Theta \vdash_L L : \Theta' \rightarrow \Theta''$, where $\Theta, \Theta', \Theta'' \subseteq \text{Meths}$ and $\Theta' \cap \Theta'' = \emptyset$. The judgment $\emptyset \vdash_L L : \Theta' \rightarrow \Theta''$ guarantees that any method occurring in L is present either in Θ' or Θ'' , that all methods in Θ' have been declared as abstract and not implemented, while all methods in Θ'' have been declared

as public and defined. Thus, $\emptyset \vdash_L L : \Theta \rightarrow \Theta'$ stands for a library in which Θ, Θ' are the abstract and public methods respectively. In this case, we also write $L : \Theta \rightarrow \Theta'$.

Remark 19. For simplicity, we do not include private methods but the same effect could be achieved by storing them in higher-order references. As we explain in the next section, references present in library definitions are de facto private to the library.

Note also that, according to our definition, sets of abstract and public methods are disjoint. However, given $m, m' \in \text{Refs}_{\theta, \theta'}$, one can define a “public abstract” method with: public m ; abstract m' ; $m = \lambda x^\theta. m' x$.

4 Semantics

The semantics of our system will be given in several stages. First, we define an operational semantics for sequential and concurrent terms that may draw methods from a function repository. We then adapt it to capture interactions of concurrent clients with libraries that do not feature abstract methods. The extended notion is then used to define contextual approximation (refinement) for arbitrary libraries. Subsequently, we introduce a trace semantics of arbitrary libraries that will be used to define higher-order notions of linearisability and, ultimately, to relate them to contextual refinement.

4.1 Library-client evaluation

Libraries, terms and clients are evaluated in environments comprising:

- A method environment \mathcal{R} , called *own-method repository*, which is a finite partial map on Meths assigning to each m in its domain, with $m \in \text{Meths}_{\theta, \theta'}$, a term of the form $\lambda y.M$ (we omit type-superscripts from bound variables for economy).
- A finite partial map $S : \text{Refs} \rightarrow (\mathbb{Z} \cup \text{Meths})$, called *store*, which assigns to each r in its domain an integer (if $r \in \text{Refs}_{\text{int}}$) or name from $\text{Meths}_{\theta, \theta'}$ (if $r \in \text{Refs}_{\theta, \theta'}$).

The evaluation rules are presented in Figure 6.

Remark 20. We shall assume that reference names used in libraries are library-private, i.e. sets of reference names used in different libraries are assumed to be disjoint. Similarly, when libraries are being used by client code, this is done on the understanding that the references available to that code do not overlap with those used by libraries. Still, for simplicity, we shall rely on a single set Refs of references in our operational rules.

First we evaluate the library to create an initial repository and store. This is achieved by the first set of rules in Figure 6, where we assume that S_{init} is empty. Note that m in the last rule is any fresh method name of the appropriate type. Thus, library evaluation produces a tuple $(\epsilon, \mathcal{R}_0, S_0)$ including a method repository and a store, which can be used as the initial repository and store for evaluating $M_1 \parallel \dots \parallel M_N$ using the (K_N) rule. We shall call the latter evaluation semantics for clients (denoted by \Longrightarrow) the *multi-threaded operational semantics*.

Reduction rules rely on evaluation contexts E , defined along with values v in the third group in Figure 6. Finally, rules for closed-term reduction (\rightarrow_t) are given in the last group, where t is the current thread index. Note that the rule for $E[\lambda x.M]$ involves the creation of a new method name m , which is used to put the function in the repository \mathcal{R} .

We define termination for clients linked with libraries that have no abstract methods.

$(L) \longrightarrow_{\text{lib}} (L, \emptyset, S_{\text{init}})$	$(r := i; B, \mathcal{R}, S) \longrightarrow_{\text{lib}} (B, \mathcal{R}, S[r \mapsto i])$
$(\text{abstract } m; L, \mathcal{R}, S) \longrightarrow_{\text{lib}} (L, \mathcal{R}, S)$	$(m = \lambda x.M; B, \mathcal{R}, S) \longrightarrow_{\text{lib}} (B, \mathcal{R} \uplus (m \mapsto \lambda x.M), S)$
$(\text{public } m; L, \mathcal{R}, S) \longrightarrow_{\text{lib}} (L, \mathcal{R}, S)$	$(r := \lambda x.M; B, \mathcal{R}, S) \longrightarrow_{\text{lib}} (B, \mathcal{R} \uplus (m \mapsto \lambda x.M), S_*)$
<hr/>	
$(M, \mathcal{R}, S) \rightarrow_t (M', \mathcal{R}', S')$	
$(M_1 \parallel \dots \parallel M_{t-1} \parallel M \parallel M_{t+1} \parallel \dots \parallel M_N, \mathcal{R}, S) \Longrightarrow (M_1 \parallel \dots \parallel M_{t-1} \parallel M' \parallel M_{t+1} \parallel \dots \parallel M_N, \mathcal{R}', S') \quad (K_N)$	
<hr/>	
$E ::= \bullet \mid E \oplus M \mid i \oplus E \mid \text{if } E \text{ then } M \text{ else } M \mid \pi_j E \mid \langle E, M \rangle \mid \langle v, E \rangle \mid mE \mid \text{let } x = E \text{ in } M \mid r := E$ $v ::= () \mid i \mid m \mid \langle v, v \rangle$	
<i>(Evaluation Contexts and Values)</i>	
<hr/>	
$(E[t_{\text{id}}], \mathcal{R}, S) \rightarrow_t (E[t], \mathcal{R}, S)$	$(E[i_1 \oplus i_2], \mathcal{R}, S) \rightarrow_t (E[i], \mathcal{R}, S) \quad (i = i_1 \oplus i_2)$
$(E[!r], \mathcal{R}, S) \rightarrow_t (E[S(r)], \mathcal{R}, S)$	$(E[\text{if } 0 \text{ then } M_1 \text{ else } M_0], \mathcal{R}, S) \rightarrow_t (E[M_0], \mathcal{R}, S)$
$(E[r := v], \mathcal{R}, S) \rightarrow_t (E[()], \mathcal{R}, S_{**})$	$(E[\text{if } i_* \text{ then } M_1 \text{ else } M_0], \mathcal{R}, S) \rightarrow_t (E[M_1], \mathcal{R}, S)$
$(E[\lambda x.M], \mathcal{R}, S) \rightarrow_t (E[m], \mathcal{R}_{**}, S)$	$(E[mv], \mathcal{R}_{**}, S) \rightarrow_t (E[M\{v/x\}], \mathcal{R}_{**}, S)$
$(E[\pi_j(v_1, v_2)], \mathcal{R}, S) \rightarrow_t (E[v_j], \mathcal{R}, S)$	$(E[\text{let } x = v \text{ in } M], \mathcal{R}, S) \rightarrow_t (E[M\{v/x\}], \mathcal{R}, S)$

Fig. 6. Evaluation rules for libraries ($\longrightarrow_{\text{lib}}$), clients (\Longrightarrow), and terms (\rightarrow_t). Here, $S_* = S[r \mapsto m]$, $S_{**} = S[r \mapsto v]$, $\mathcal{R}_* = \mathcal{R} \uplus (m \mapsto \lambda x.M)$; and $i_* \neq 0$, $\mathcal{R}_{**}(m) = \lambda x.M$.

Definition 21. Let $L : \emptyset \rightarrow \Theta'$ and $\Theta' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$.⁴ We say that $M_1 \parallel \dots \parallel M_N$ *terminates with linked library* L if

$$(M_1 \parallel \dots \parallel M_N, \mathcal{R}_0, S_0) \Longrightarrow^* ((), \mathcal{R}, S)$$

for some \mathcal{R}, S , where $(L) \longrightarrow_{\text{lib}}^* (\epsilon, \mathcal{R}_0, S_0)$. Then we write $\text{link } L \text{ in } (M_1 \parallel \dots \parallel M_N) \Downarrow$.

We shall build a notion of contextual approximation of libraries on top of termination: one library will be said to approximate another if, whenever the former terminates when composed with any parameter library and client, so does the latter.

There are several ways of composing libraries. Here we will be considering the notions of *union* and *sequencing*. The latter is derived from the former with the aid of a third construct, called *hiding*. Below, we denote a library L as $L = D; B$, where D contains all the (public/abstract) method declarations of L , and B is its method block.

Definition 22. [Library union, hiding, sequencing] Let $L_1 : \Theta_1 \rightarrow \Theta_2$ be a library of the form $D_1; B_1$.

- Given library $L_2 : \Theta'_1 \rightarrow \Theta'_2 (= D_2; B_2)$ which accesses disjoint parts of the store from L_1 and such that $\Theta_2 \cap \Theta'_2 = \emptyset$, we define the *union* of L_1 and L_2 as:

$$L_1 \cup L_2 : (\Theta_1 \cup \Theta'_1) \setminus (\Theta_2 \cup \Theta'_2) \rightarrow \Theta_2 \cup \Theta'_2 = (D_1; B_1) \cup (D_2; B_2) = D'_1; D'_2; B_1; B_2$$

where D'_1 is D_1 with any “abstract m ” declaration removed for $m \in \Theta'_2$; dually for D'_2 .

⁴ Recall our convention (Remark 20) that L and M_1, \dots, M_N must access disjoint parts of the store. Terms M_1, \dots, M_N can share reference names, though.

- Given some $\Theta = \{m_1, \dots, m_n\} \subseteq \Theta_2$, we define the Θ -hiding of L_1 as:

$$L_1 \setminus \Theta : \Theta_1 \rightarrow (\Theta_2 \setminus \Theta) = (D_1; B_1) \setminus \Theta = D'_1; B'_1\{!r_1/m_1\} \dots \{!r_n/m_n\}$$

where D'_1 is D_1 without public m declarations for $m \in \Theta$ and, for each i , r_i is a fresh reference matching the type of m_i , and B'_1 is obtained from B_1 by replacing each definition $m_i = \lambda x.M$ by $r_i := \lambda x.M$.

The sequencing of $L' : \emptyset \rightarrow \Theta_1, \Theta'$ with L_1 is: $L'; L_1 : \emptyset \rightarrow \Theta_2, \Theta' = (L' \cup L_1) \setminus \Theta_1$.

Thus, the union of two libraries L_1 and L_2 as above corresponds to merging their code and removing any abstract method declarations for methods defined in the union. On the other hand, the hiding of a public method simply renders it private via the use of references. These notions are used in defining contextual notions for libraries, that is, notions that require quantification over all possible contexts.

Definition 23. Given $L_1, L_2 : \Theta \rightarrow \Theta'$, we say that L_1 *contextually approximates* L_2 , written $L_1 \sqsubseteq L_2$, if for all $L' : \emptyset \rightarrow \Theta, \Theta''$ and $\Theta', \Theta'' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$, if link $L'; L_1$ in $(M_1 \parallel \dots \parallel M_N) \Downarrow$ then link $L'; L_2$ in $(M_1 \parallel \dots \parallel M_N) \Downarrow$. In this case, we also say that L_2 *contextually refines* L_1 .

Note that, according to this definition, the parameter library L' may communicate directly with the client terms through a common interface Θ'' . We shall refer to this case as the *general* case. Later on, we shall also consider more restrictive testing scenarios in which this possibility of explicit communication is removed. Moreover, from the disjointness conditions in the definitions of sequencing and linking we have that L_i, L' and $M_1 \parallel \dots \parallel M_N$ access pairwise disjoint parts of the store.

Remark 24. Our definition of contextual approximation models communication between the client and the parameter library explicitly through the shared interface Θ'' . This is different in style (but not in substance) from [3], where the presence of public abstract methods inside the tested library provides such a communication channel.

4.2 Trace semantics

Building on the earlier operational semantics, we next introduce a trace semantics of libraries, in the spirit of game semantics [1]. As mentioned in Section 2, the behaviour of a library will be represented as an exchange of moves between two players called O and P , representing the library (P) and the corresponding context (O) respectively. The context consists of the client of the library as well as the parameter library, with an index on each move specifying which of them is involved in the move (\mathcal{K} or \mathcal{L} respectively).

In contrast to the semantics of the previous section, we will handle scenarios in which methods need not be present in the repository \mathcal{R} . Calls to such undefined methods will be represented by labelled transitions – calls to the context made on behalf of the library (P). The calls can later be responded to with labelled transitions corresponding to returns, made by the context (O). On the other hand, O will be able to invoke methods in \mathcal{R} , which will also be represented through suitable labels. Because we work in a higher-order setting, calls and returns made by both players may involve methods as arguments or results. Such methods also become available for future calls: function

arguments/results supplied by P are added to the repository and can later be invoked by O , while function arguments/results provided by O can be queried in the same way as abstract methods.

After giving semantics to libraries, we shall also define a semantics for **contexts**, i.e. clients paired with parameter libraries where the main library is missing. More precisely, given a parameter library $L' : \emptyset \rightarrow \Theta, \Theta'$ and client $\Theta', \Theta'' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$, we will define the semantics of $M_1 \parallel \dots \parallel M_N$ when paired with L' . In such a scenario, the roles of O and P will be reversed: PK will own moves played by the client, PL will be the parameter library, while O will correspond to the missing main library (OK when interacting with the client, and OL when talking to the parameter library).

Recall the notions of history (Def. 2) and history complementation (Def. 15). We next define a dual notion of history that is used for assigning semantics to contexts.

Definition 25. The set of **co-histories** over $\Theta \rightarrow \Theta'$ is: $\mathcal{H}_{\Theta, \Theta'}^{co} = \{\bar{h} \mid h \in \mathcal{H}_{\Theta, \Theta'}\}$.

We shall range over $\mathcal{H}_{\Theta, \Theta'}^{co}$ again using variables h, s . We can show the following.

Lemma 26. – For all $h \in \mathcal{H}_{\Theta, \Theta'}$, we have $h \upharpoonright \mathcal{L} \in \mathcal{H}_{\emptyset, \Theta}^{co}$ and $h \upharpoonright \mathcal{K} \in \mathcal{H}_{\emptyset, \Theta'}$.
– For all $h \in \mathcal{H}_{\Theta, \Theta'}^{co}$, we have $h \upharpoonright \mathcal{L} \in \mathcal{H}_{\emptyset, \Theta}$ and $h \upharpoonright \mathcal{K} \in \mathcal{H}_{\emptyset, \Theta'}$.

The trace semantics will utilise configurations that carry more components than the previous semantics, in order to compensate for the fact that we need to keep track of the evaluation history that is currently processed, as well as the method names that have been passed between O and P . We define two kinds of configurations:

O-configurations $(\mathcal{E}, -, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ and *P-configurations* $(\mathcal{E}, M, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$,

where the component \mathcal{E} is an *evaluation stack*, that is, a stack of the form $[X_1, X_2, \dots, X_n]$ with each X_i being either an evaluation context or a method name. On the other hand, $\mathcal{P} = (\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{K}})$ with $\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{K}} \subseteq \text{dom}(\mathcal{R})$ being sets of *public* method names, and $\mathcal{A} = (\mathcal{A}_{\mathcal{L}}, \mathcal{A}_{\mathcal{K}})$ is a pair of sets of *abstract* method names. \mathcal{P} will be used to record all the method names produced by P and passed to O : those passed to OK are stored in $\mathcal{P}_{\mathcal{K}}$, while those leaked to OL are kept in $\mathcal{P}_{\mathcal{L}}$. Inside \mathcal{A} , the story is the opposite one: $\mathcal{A}_{\mathcal{K}}$ ($\mathcal{A}_{\mathcal{L}}$) stores the method names produced by OK (resp. OL) and passed to P . Consequently, the sets of names stored in $\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{K}}, \mathcal{A}_{\mathcal{L}}, \mathcal{A}_{\mathcal{K}}$ will always be disjoint.

Given a pair \mathcal{P} as above and a set $Z \subseteq \text{Meths}$, we write $\mathcal{P} \cup_{\mathcal{K}} Z$ for the pair $(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{K}} \cup Z)$. We define $\cup_{\mathcal{L}}$ in a similar manner, and extend it to pairs \mathcal{A} as well. Moreover, given \mathcal{P} and \mathcal{A} , we let $\phi(\mathcal{P}, \mathcal{A})$ be the set of *fresh* method names for \mathcal{P}, \mathcal{A} : $\phi(\mathcal{P}, \mathcal{A}) = \text{Meths} \setminus (\mathcal{P}_{\mathcal{L}} \cup \mathcal{P}_{\mathcal{K}} \cup \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{K}})$.

We next give the rules generating the trace semantics. Note that the rules are parameterised by $Y \in \{\mathcal{K}, \mathcal{L}\}$. This parameter will play a role in our treatment of the encapsulated case, as it allows us to track the labels related to interactions with the client and the parameter library respectively. In all of the rules below, whenever we write $m(v)$ or $m(v')$, we assume that the type of v matches the argument type of m .

Internal rule First we embed the internal rules, introduced earlier: if $(M, \mathcal{R}, S) \rightarrow_t (M', \mathcal{R}', S')$ and $\text{dom}(\mathcal{R}' \setminus \mathcal{R})$ consists of names that do not occur in \mathcal{E}, \mathcal{A} , we have:

$$(\mathcal{E}, M, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \rightarrow_t (\mathcal{E}, M', \mathcal{R}', \mathcal{P}, \mathcal{A}, S') \quad (\text{INT})$$

This includes internal method calls (i.e. $(E[mv], \mathcal{R}, S) \rightarrow_t \dots$ with $m \in \text{dom}(\mathcal{R})$).

P calls In the next family of rules, the library (P) calls one of its abstract methods (either the original ones or those acquired via interaction). Thus, the rule applies to $m \in \mathcal{A}_Y$.

$$(\mathcal{E}, E[mv], \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\text{call } m(v')_{PY}}_t (m :: E :: \mathcal{E}, -, \mathcal{R}', \mathcal{P}', \mathcal{A}, S) \quad (\text{PQY})$$

If v does not contain any method names then $v' = v$, $\mathcal{R}' = \mathcal{R}$, $\mathcal{P}' = \mathcal{P}$. Otherwise, if v contains the (pairwise distinct) names m_1, \dots, m_k , a fresh name $m'_i \in \phi(\mathcal{P}, \mathcal{A})$ is created for each method name m_i (for future reference to the method), and replaced for m_i in v . That is, $v' = v\{m'_i/m_i \mid 1 \leq i \leq k\}$. We must also have that the m'_i 's are pairwise distinct (the rule fires for any such m'_i 's), and also $\mathcal{R}' = \mathcal{R} \uplus \{m'_i \mapsto \lambda x. m_i x \mid 1 \leq i \leq k\}$ and $\mathcal{P}' = \mathcal{P} \cup_Y \{m'_1, \dots, m'_k\}$.

P returns Analogously, the library (P) may return a result to an earlier call made by the context. This rule is applicable provided $m \in \mathcal{P}_Y$.

$$(m :: \mathcal{E}, v, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\text{ret } m(v')_{PY}}_t (\mathcal{E}, -, \mathcal{R}', \mathcal{P}', \mathcal{A}, S) \quad (\text{PAY})$$

$v', \mathcal{R}', \mathcal{P}'$ are subject to the same constraints as in (PQY).

O calls The remaining rules are dual and represent actions of the context. Here the context calls a public method: either an original one or one that has been made public later (by virtue of having been passed to the context by the library). Here we require $m \in \mathcal{P}_Y$ and $\mathcal{R}(m) = \lambda x. M$.

$$(\mathcal{E}, -, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\text{call } m(v)_{OY}}_t (m :: \mathcal{E}, M\{v/x\}, \mathcal{R}, \mathcal{P}, \mathcal{A}', S) \quad (\text{OQY})$$

If v contains the names $m_1, \dots, m_k \in \text{Meths}$, it must be the case that $m_i \in \phi(\mathcal{P}, \mathcal{A})$, for each i , and $\mathcal{A}' = \mathcal{A} \cup_Y \{m_1, \dots, m_k\}$.

O returns Finally, we have rules corresponding to values being returned by the context in response to calls made by the library. In this case we insist on $m \in \mathcal{A}_Y$.

$$(m :: E :: \mathcal{E}, -, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\text{ret } m(v)_{OY}}_t (\mathcal{E}, E[v], \mathcal{R}, \mathcal{P}, \mathcal{A}', S) \quad (\text{OAY})$$

As in the previous case, if $m \in \text{Meths}$ is present in v then we need $m \in \phi(\mathcal{P}, \mathcal{A})$ and \mathcal{A}' is calculated in the same way.

Finally, we extend the trace semantics to a concurrent setting where a fixed number of N -many threads run in parallel. Each thread has separate evaluation stack and term components, which we write as $\mathcal{C} = (\mathcal{E}, X)$ (where X is a term or “-”). Thus, a configuration now is of the following form, and we call it an N -configuration:

$$(\mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_N, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$$

where, for each i , $\mathcal{C}_i = (\mathcal{E}_i, X_i)$ and $(\mathcal{E}_i, X_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ is a sequential configuration. We shall abuse notation a little and write $(\mathcal{C}_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ for $(\mathcal{E}_i, X_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$. Also, below we write $\vec{\mathcal{C}}$ for $\mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_N$ and $\vec{\mathcal{C}}[i \mapsto \mathcal{C}'] = \mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_{i-1} \parallel \mathcal{C}' \parallel \mathcal{C}_{i+1} \parallel \dots \parallel \mathcal{C}_N$.

The concurrent traces are produced by the following two rules with the proviso that the names freshly produced internally in (PINT) are fresh for the whole of \vec{C} .

$$\frac{(\mathcal{C}_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \rightarrow_i (\mathcal{C}', \mathcal{R}', \mathcal{P}, \mathcal{A}, S')}{(\vec{C}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \Longrightarrow (\vec{C}[i \mapsto \mathcal{C}'], \mathcal{R}', \mathcal{P}, \mathcal{A}, S')} \text{ (PINT)}$$

$$\frac{(\mathcal{C}_i, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{x_{XY}}_i (\mathcal{C}', \mathcal{R}', \mathcal{P}', \mathcal{A}', S')}{(\vec{C}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{(i, x)_{XY}} (\vec{C}[i \mapsto \mathcal{C}'], \mathcal{R}', \mathcal{P}', \mathcal{A}', S')} \text{ (PEXT)}$$

We can now define the trace semantics of a library L . We call a configuration component \mathcal{C}_i **final** if it is in one of the following forms:

$$\mathcal{C}_i = ([], -) \text{ or } \mathcal{C}_i = ([], ())$$

for O - and P -configurations respectively. We call $(\vec{C}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ final just if $\vec{C} = \mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_N$ and each \mathcal{C}_i is final.

Definition 27. For each $L : \Theta \rightarrow \Theta'$, we define the N -trace semantics of L to be:

$$\llbracket L \rrbracket_N = \{ s \mid (\vec{C}_0, \mathcal{R}_0, (\emptyset, \Theta'), (\Theta, \emptyset), S_0) \xrightarrow{s}^* \rho \wedge \rho \text{ final} \}$$

where $\vec{C}_0 = ([], -) \parallel \dots \parallel ([], -)$ and $(L) \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}_0, S_0)$.

For economy, in the sequel we might be dropping the index N from $\llbracket L \rrbracket_N$.

We conclude this section by providing a semantics for library contexts. Recall that in the definition of contextual approximation the library $L : \Theta \rightarrow \Theta'$ is deployed in a context consisting of a parameter library $L' : \emptyset \rightarrow \Theta, \Theta''$ and a concurrent composition of client threads $\Theta', \Theta'' \vdash M_i : \text{unit}$ ($i = 1, \dots, N$). This context makes internal use of methods defined in the parameter library, while it recurs to the trace system for the methods in Θ' . At the same time, the context provides the methods in Θ in the trace system. We shall write link $L'; -$ in $(M_1 \parallel \dots \parallel M_N)$, or simply C , to refer to such contexts. We give the following semantics to contexts.

Definition 28. Let $\Theta', \Theta'' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$ and $L' : \emptyset \rightarrow \Theta, \Theta''$. We define the semantics of the context formed by L' and M_1, \dots, M_N to be:

$$\llbracket \text{link } L'; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket = \{ s \mid (\vec{C}_0, \mathcal{R}_0, (\Theta, \emptyset), (\emptyset, \Theta'), S_0) \xrightarrow{s}^* \rho \wedge \rho \text{ final} \}$$

where $(L') \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}_0, S_0)$ and $\vec{C}_0 = ([], M_1) \parallel \dots \parallel ([], M_N)$.

Lemma 29. For any $L : \Theta \rightarrow \Theta'$, $L' : \emptyset \rightarrow \Theta, \Theta''$ and $\Theta', \Theta'' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$ we have $\llbracket L \rrbracket_N \subseteq \mathcal{H}_{\Theta, \Theta'}$ and $\llbracket \text{link } L'; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket \subseteq \mathcal{H}_{\Theta, \Theta'}^{\text{co}}$.

Proof. The relevant sequences of moves are clearly alternating and well-bracketed, when projected on single threads, because the LTS is bipartite (O - and P -configurations) and separate evaluation stacks control the evolution in each thread. Other conditions for histories follow from the partitioning of names into $\mathcal{A}_K, \mathcal{A}_L, \mathcal{P}_K, \mathcal{P}_L$ and suitable initialisation: Θ, Θ' are inserted into $\mathcal{A}_L, \mathcal{P}_K$ respectively (for $\llbracket L \rrbracket$) and into $\mathcal{P}_L, \mathcal{A}_K$ for $\llbracket C \rrbracket$. \square

5 Examples

We now revisit the example libraries from Section 2 and show they each linearise to their respective specification, according to the relevant notion of linearisability (general/encapsulated/relational).

5.1 Multiset examples

Recall the multiset library L_{mset} and the specification A_{mset} of Example 6 and Figure 2. We show that $L_{\text{mset}} \sqsubseteq A_{\text{mset}}$. More precisely, taking an arbitrary history $h \in \llbracket L_{\text{mset}} \rrbracket$ we show that h can be rearranged using \triangleleft_{PO}^* to match an element of A_{mset} . We achieve this by identifying, for each O -move $(t, x)_O$ and its following P -move $(t, x')_P$ in h , a *linearisation point* between them, i.e. a place in h to which $(t, x)_O$ can be moved right and to which $(t, x')_P$ can be moved left so that they become consecutive and, moreover, the resulting history is still produced by L_{mset} . After all these rearrangements, we obtain a sequential history \hat{h} such that $h \sqsubseteq \hat{h}$ and \hat{h} is also produced by L_{mset} . It then suffices to show that $\hat{h} \in A_{\text{mset}}$.

Lemma 30 (Multiset). L_{mset} linearises to A_{mset} .

Proof. Taking an arbitrary $h \in \llbracket L_{\text{mset}} \rrbracket$, we demonstrate the linearisation points for pairs of (O, P) moves in h , by case analysis on the moves (we drop \mathcal{K} indices from moves as they are ubiquitous). Let us assume that h has been generated by a sequence $\rho_1 \Rightarrow \rho_2 \Rightarrow \dots \Rightarrow \rho_k$ of atomic transitions and that the variable F of L_{mset} is instantiated with the reference r_F . Line numbers used below will refer to Figure 2.

1. $h = \dots (t, \text{call } cnt(i))_O s (t, \text{ret } cnt(j))_P \dots$. Here the linearisation point is the configuration ρ_i that dereferences r_F as per line 6 in L_{mset} (the $!F$ expression).
2. $h = \dots (t, \text{call } upd(i, m))_O s (t, \text{call } m(j))_P \dots$. The linearisation point is the dereferencing of r_F in line 8.
3. $h = \dots (t, \text{ret } m(j'))_O s (t, \text{ret } upd(|j'|))_P \dots$. The linearisation point is the update to r_F in line 14.
4. $h = \dots (t, \text{ret } m(j'))_O s (t, \text{call } m(j''))_P \dots$. The linearisation point is the dereferencing of r_F in line 12.

Each of the linearisation points above specifies a PO -rearrangement of moves. For instance, for $h = s_0 (t, \text{call } cnt(i))_O s (t, \text{ret } cnt(j))_P s'$, let $s = s_1 s_2$ where $s_0 s_1$ is the prefix of h produced by $\rho_1 \Rightarrow \rho_2 \Rightarrow \dots \Rightarrow \rho_i$. The rearrangement of h is then the history $\hat{h} = s_0 s_1 (t, \text{call } upd(i, m))_O (t, \text{call } m(j))_P s_2 s'$. We thus obtain $h \triangleleft_{PO}^* \hat{h}$.

The selection of linearisation points is such that it guarantees that $\hat{h} \in \llbracket L_{\text{mset}} \rrbracket$. E.g. in case 1, the transitions occurring in thread t between the configuration that follows $(t, \text{call } cnt(i))_O$ and ρ_i do not access r_F . Hence, we can postpone them and fire them in sequence just ρ_i . After ρ_{i+1} and until $(t, \text{ret } cnt(j))_P$ there is again no access of r_F in t and we can thus bring forward the corresponding transitions just after ρ_{i+1} . Similar reasoning applies to case 2. In case 4, we reason similarly but also take into account that rendering the acquisition of the lock by t atomic is sound (i.e. the semantics can produce the rearranged history). Case 3 is similar, but we also use the fact that the access to r_F in lines 11-16 is inside the lock, and hence postponing dereferencing (line 12) to occur in sequence before update (line 14) is sound.

Now, any transition sequence α which produces \hat{h} (in $\llbracket L_{\text{mult}} \rrbracket$) can be used to derive an extended history $h^\circ \in A_{\text{mult}}^\circ$, by attaching to each move in \hat{h} the multiset represented in the configuration that produces the move (ρ produces the move x if $\rho \xrightarrow{x} \rho'$ in α). By projection we then obtain $\hat{h} \in A_{\text{mult}}$. \square

On the other hand, the multiset library of Example 14 and Figure 3 requires encapsulation in order to linearise (cf. Example 14).

Lemma 31 (Parameterised multiset). L_{mset2} *enc-linearises to* A_{mset2} .

Proof. Again, we identify linearisation points, this time for given $h \in \llbracket L_{\text{mult2}} \rrbracket_{\text{enc}}$. For cases 1-4 as above we reason as in Lemma 30. For *upd_enc* we have the following case.

$$h = s(t, \text{call } \text{upd_enc}(i))_{OK} s_1(t, \text{call } \text{foo}(j))_{PL} s_2(t, \text{ret } \text{foo}(j'))_{OL} s_3(t, \text{ret } \text{upd_enc}(|j'|))_{PK} \dots$$

Here, we need a linearisation point for all four moves above. We pick this to be the point corresponding to the update of the multiset reference F on line 24 (Figure 3). We now transform h to \hat{h} so that the four moves become consecutive, in two steps:

- Let us write s_3 as $s_3 = s_3^1 s_3^2$, where the split is at the linearisation point. Since the lock is constantly held by thread t in $s_2 s_3^1$, there can be no calls or returns to *foo* in $s_2 s_3^1$. Hence, all moves in $s_2 s_3^1$ are in component \mathcal{K} and can be transposed with the \mathcal{L} -moves above, using \diamond^* , to obtain $h' = s(t, \text{call } \text{upd_enc}(i))_{OK} s_1 s_2 s_3^1(t, \text{call } \text{foo}(j))_{PL} (t, \text{ret } \text{foo}(j'))_{OL} s_3^2(t, \text{ret } \text{upd_enc}(|j'|))_{PK} \dots$
- Next, by *PO*-rearrangement we obtain $\hat{h} = s s_1 s_2 s_3^1(t, \text{call } \text{upd_enc}(i))_{OK} (t, \text{call } \text{foo}(j))_{PL} (t, \text{ret } \text{foo}(j'))_{OL} (t, \text{ret } \text{upd_enc}(|j'|))_{PK} s_3^2 \dots$. Thus, $h(\triangleleft_{PO} \cup \diamond)^* \hat{h}$.

To prove that $\hat{h} \in A_{\text{mult2}}$ we work as in Lemma 30, i.e. via showing that $\hat{h} \in \llbracket L_{\text{mult2}} \rrbracket_{\text{enc}}$. For the latter, we rely on the fact that the linearisation point was taken at the reference update point (so that any dereferencings from other threads are preserved), and that the dereferences of lines 22 and 23 are within the same lock as the update. \square

5.2 Flat combining

Recall the libraries L_{fc} and L_{spec} from Example 17 and Figure 4. We shall investigate the impact of introducing higher-order types to the flat combining algorithm, which will lead to several surprising discoveries. First of all, let us observe that, even if $\theta_i = \theta'_i = \text{int}$, higher-order interactions of both L_{spec} and L_{fc} with a client and parameter library according to Definition 23 (general case) may lead to deadlock. In this case, a client can communicate with the parameter library (via Θ'') and, for example, he could supply it with a function that calls a public method of the library, say, m'_i . That function could then be used to implement m_i and, consequently, a client call to m'_i would result in lock acquisition, then a call to m_i , which would trigger another call to m'_i and an attempt to acquire the same lock, while m_i cannot return (cf. Example 4).

Deadlock can also arise in the encapsulated case (Definition 37) if the library contains a public method, say m'_i , with functional arguments. Then the client can pass a function that calls m'_i as an argument to m'_i and, if the abstract method m_i subsequently calls the argument, deadlock would follow in the same way as before. Correspondingly, in these cases there exist sequences of transitions induced by our transition system that cannot

be extended to a history, e.g. for $\theta_i = \text{unit} \rightarrow \text{unit}$ this happens after $(1, \text{call } m'(v))_{OK}$ $(1, \text{call } m(v')_{PL})$ $(1, \text{call } v'())_{OL}$ $(1, \text{call } v())_{PK}$ $(1, \text{call } m'(v''))_{OK}$. Consequently, the protocol should not really be used in an unrestricted higher-order setting.

However, the phenomenon described above cannot be replicated in the encapsulated case provided the argument types are ground ($\theta_i = \text{unit}, \text{int}$). In this case, without imposing any restrictions on the result types θ'_i , we shall show that $L_{fc} \sqsubseteq_{\mathcal{R}} L_{\text{spec}}$, where \mathcal{R} stands for thread-name invariance. Note that this is a proper extension of the result in [3], where θ'_i had to be equal to unit or int . It is really necessary to use the finer notion of $\sqsubseteq_{\mathcal{R}}$ here, as we do not have $L_{fc} \sqsubseteq_{\text{enc}} L_{\text{spec}}$ (a parameter library that is sensitive to thread identifiers may return results that allow one to detect that a request has been handled by a combiner thread which is different from the original one).

Lemma 32 (Flat combining). *Let $\Theta = \{m_i \in \text{Meths}_{\text{int}, \theta'_i} \mid 1 \leq i \leq k\}$ and $\Theta' = \{m'_i \in \text{Meths}_{\text{int}, \theta'_i} \mid 1 \leq i \leq k\}$ be such that $\Theta \cap \Theta' = \emptyset$. Let \mathcal{R} consist of pairs $(h_1, h_2) \in \mathcal{H}_{\emptyset, \Theta} \times \mathcal{H}_{\emptyset, \Theta'}$ that are identical once thread identifiers are ignored. Then $L_{fc} \sqsubseteq_{\mathcal{R}} L_{\text{spec}}$.*

Proof. Observe that histories from $\llbracket L_{\text{spec}} \rrbracket_{\text{enc}}$ feature threads built from segments of one of the three forms (we suppress integer arguments for economy):

- $(t, \text{call } m'_i)_{OK} (t, \text{call } m_i)_{PL} (t, \text{ret } m_i(v))_{OL} (t, \text{ret } m'_i(v'))_{PK}$, or
- $(t', \text{call } w(v))_{OY} (t', \text{call } w'(v'))_{PY'}$ for $Y \neq Y'$, where w is a name introduced in an earlier move $(t'', x)_{PY}$ and w' is a corresponding name introduced by the move preceding $(t'', x)_{PY}$, or
- $(t', \text{ret } w'(v''))_{OY'} (t', \text{ret } w(v'''))_{PY}$ such that a segment $(t', \text{call } w(v))_{OY} (t', \text{call } w'(v'))_{PY'}$ already occurred earlier.

We shall refer to moves in the second and third kind of segments as *inspection moves* and write ϕ to refer to sequences built exclusively from such sequences (we will use $\phi^{\mathcal{K}}$ and $\phi^{\mathcal{L}}$ if we want to stress that the moves are exclusively tagged with \mathcal{K} or \mathcal{L}). Let us write \mathcal{X} for the subset of $\llbracket L_{\text{spec}} \rrbracket_{\text{enc}}$ containing plays of the form:

$$\begin{aligned} & (t_0, \text{call } m'_{i_0}) (t_0, \text{call } m_{i_0}) (t_0, \text{ret } m_{i_0}(v_0)) (t_0, \text{ret } m'_{i_0}(v'_0)) \phi_0 \\ & (t_1, \text{call } m'_{i_1}) (t_1, \text{call } m_{i_1}) \phi_1 (t_1, \text{ret } m_{i_1}(v_1)) (t_1, \text{ret } m'_{i_1}(v'_1)) \phi_2 \\ & \dots (t_k, \text{call } m'_{i_k}) (t_k, \text{call } m_{i_k}) \phi_{2k-1} (t_k, \text{ret } m_{i_k}(v_k)) (t_k, \text{ret } m'_{i_k}(v'_k)) \phi_{2k} \end{aligned}$$

where each ϕ_i contains moves alternating between O and P .

Let $h_1 \in \llbracket L_{fc} \rrbracket_{\text{enc}}$. Threads in h_1 are built from blocks of the shapes:

$$\begin{aligned} & (t, \text{call } m'_i)_{OK} ((t, \text{call } m_j)_{PL} (t, \text{ret } m_j(v))_{OL})^* (t, \text{ret } m'_i(v'))_{PK} \\ & \text{or } (t', \text{call } w(v))_{OY} (t', \text{call } w'(v'))_{PY'} \text{ or } (t', \text{ret } w'(v''))_{OY} (t', \text{ret } w(v'''))_{PY'}. \end{aligned}$$

In the first case, the j 's and v 's are meant to represent different values in each iteration. In the second kind of block, w needs to be introduced earlier by some $(t'', x)_{PY}$ move and w' is then a name introduced by the preceding move. For the third kind, an earlier calling sequence of the second kind must exist in the same thread.

Note that, due to locking and sequentiality of loops, $h_1 \upharpoonright \mathcal{L}$ takes the shape:

$$\begin{aligned} & (t_0, \text{call } m_{i_0})_P (t_0, \text{ret } m_{i_0}(v_0))_O \phi_0^{\mathcal{L}} (t_1, \text{call } m_{i_1})_P \phi_1^{\mathcal{L}} (t_1, \text{ret } m_{i_1}(v_1))_O \phi_2^{\mathcal{L}} \\ & \dots (t_k, \text{call } m_{i_k})_P \phi_{2k-1}^{\mathcal{L}} (t_k, \text{ret } m_{i_k}(v_k))_O \phi_{2k}^{\mathcal{L}} \end{aligned}$$

Each segment $S_j = (t_j, \text{call } m_{i_j}) \phi_{2j-1}^{\mathcal{L}} (t_j, \text{ret } m_{i_j}(v_j))$ in $h_1 \upharpoonright \mathcal{L}$ must be preceded (in h_1) by a corresponding public call $(t'_j, \text{call } m'_{i_j})_{O\mathcal{K}}$ and followed by a matching return $(t'_j, \text{ret } m'_{i_j}(v_j))_{P\mathcal{K}}$, where t'_j need not be equal to t_j . Note that there can be no other moves from t'_j separating the two moves in $h_1 \upharpoonright \mathcal{K}$.

Let h'_1 be obtained from h_1 via the following operations around each segment S_j :

- move $(t'_j, \text{call } m'_{i_j})$ right to precede $(t_j, \text{call } m_{i_j})$,
- move $(t'_j, \text{ret } m'_{i_j}(v'_j))$ left to follow $(t_j, \text{ret } m_{i_j}(v_j))$,
- rename $(t_j, \text{call } m_{i_j})_{P\mathcal{L}}(t_j, \text{ret } m_{i_j}(v_j))_{O\mathcal{L}}$ to $(t'_j, \text{call } m_{i_j})_{P\mathcal{L}}(t'_j, \text{ret } m_{i_j}(v_j))_{O\mathcal{L}}$.

We stress that the changes are to be performed simultaneously for each segment S_j . The rearrangements result in a library history, because they bring forward the points at which various v_j, v'_j have been introduced and, thus, inspection moves are legal. Then we have $h_1 \sqsubseteq_{\mathcal{R}} h'_1$, i.e. $(\overline{h_1} \upharpoonright \mathcal{L})\mathcal{R}(\overline{h'_1} \upharpoonright \mathcal{L})$ and $(h_1 \upharpoonright \mathcal{K}) \triangleleft_{PO}^* (h'_1 \upharpoonright \mathcal{K})$. The former follows from the multiple renaming of the moves originally tagged with t_j to t'_j and the fact that their order in $h_1 \upharpoonright \mathcal{L}$ is unaffected. The latter holds, because O -moves move right and P -moves move left past other moves in $h_1 \upharpoonright \mathcal{K}$ that are not from the same thread. To conclude, we show that there exists $h_2 \in \mathcal{X}$ such that $(\overline{h'_1} \upharpoonright \mathcal{L}) = (\overline{h_2} \upharpoonright \mathcal{L})$ and $(h'_1 \upharpoonright \mathcal{K}) \triangleleft_{PO}^* (h_2 \upharpoonright \mathcal{K})$. We can obtain h_2 by rearranging inspection moves in different threads of h'_1 so that they alternate between O and P . Since the inspection moves come in pairs this can simply be done by bringing the paired moves next to each other. Because one of them is always from \mathcal{K} and the other from \mathcal{L} , this can be achieved by rearranging moves in $h'_1 \upharpoonright \mathcal{K}$ only: if the O -move is from \mathcal{K} it can be moved to the right, otherwise the P -move from \mathcal{K} can be moved left. Then we have $h_2 \in \mathcal{X} \subseteq \llbracket L_{\text{spec}} \rrbracket_{\text{enc}}$ with $h_1 \sqsubseteq_{\mathcal{R}} h_2$, as required. \square

6 Correctness

In this section we prove that the linearisability notions we introduce are correct: linearisability implies contextual approximation. The approach is based on showing that, in each case, the semantics of contexts is saturated relatively to conditions that are dual to linearisability. Hence, linearising histories does not alter the observable behaviour of a library. We start by presenting two compositionality theorems on the trace semantics, which will be used for relating library and context semantics.

6.1 Compositionality

The semantics we defined is compositional in the following ways:

- To compute the semantics of a library L inside a context C , it suffices to compose the semantics of C with that of L , for a suitable notion of context-library composition $(\llbracket C \rrbracket \otimes \llbracket L \rrbracket)$.
- To compute the semantics of a union library $L_1 \cup L_2$, we can compose the semantics of L_1 and L_2 , for a suitable notion of library-library composition $(\llbracket L_1 \rrbracket \otimes \llbracket L_2 \rrbracket)$.

The above are proven using bisimulation techniques for connecting syntactic and semantic compositions, and are presented in Appendices C and D respectively.

The latter correspondence is used in Appendix E for proving that linearisability is a congruence for library composition. From the former correspondence we obtain the following result, which we shall use for showing correctness.

Theorem 33. Let $L : \Theta \rightarrow \Theta'$, $L' : \emptyset \rightarrow \Theta, \Theta''$ and $\Theta', \Theta'' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$, with L, L' and $M_1; \dots; M_N$ accessing pairwise disjoint parts of the store. Then:

$$\text{link } L'; L \text{ in } (M_1 \parallel \dots \parallel M_N) \Downarrow \iff \exists h \in \llbracket L \rrbracket_N. \bar{h} \in \llbracket \text{link } L'; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket$$

6.2 General linearisability

Recall the general notion of linearisability defined in Section 2.2, which is based on move-reorderings inside histories.

In Def.s 27 and 28 we have defined the trace semantics of libraries and contexts. The semantics turns out to be closed under \triangleleft_{OP}^* .

Lemma 34 (Saturation). Let $X = \llbracket L \rrbracket$ (Def. 27) or $X = \llbracket \text{link } L'; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket$ (Def. 28). Then if $h \in X$ and $h \triangleleft_{OP}^* h'$ then $h' \in X$.

Proof. Recall that the same labelled transition system underpins the definition of X in either case. We make several observations about the single-threaded part of that system.

- The store is examined and modified only during ϵ -transitions.
- The only transition possible after a P -move is an O -move. In particular, it is never the case that a P -move is separated from the following O -move by an ϵ -transition.

Let us now consider the multi-threaded system and $t \neq t'$.

- Suppose $\rho \xrightarrow{(t', m')_P} \rho_1 \xrightarrow{\epsilon^*} \rho_2 \xrightarrow{(t, m)} \rho_3$. Then the $(t', m')_P$ -transition can be delayed inside t' until after (t, m) , i.e. $\rho \xrightarrow{\epsilon^*} \rho'_1 \xrightarrow{(t, m)} \rho'_2 \xrightarrow{(t', m')_P} \rho_3$ for some ρ'_1, ρ'_2 . This is possible because the $((t', m')_P$ -labelled) transition does not access or modify the store, and none of the ϵ -transitions distinguished above can be in t' , thanks to our earlier observations about the behaviour of the single-threaded system.
- Analogously, suppose $\rho \xrightarrow{(t', m')_O} \rho_1 \xrightarrow{\epsilon^*} \rho_2 \xrightarrow{(t, m)_O} \rho_3$. Then the $(t, m)_O$ -transition can be brought forward, i.e. $\rho \xrightarrow{(t, m)_O} \rho'_1 \xrightarrow{(t', m')_O} \rho'_2 \xrightarrow{\epsilon^*} \rho_3$, because it does not access or modify the store and the preceding ϵ -transitions cannot be from t . \square

This, along with the fact that $h_1 \triangleleft_{XX'} h_2 \iff h_2 \triangleleft_{X'X} h_1 \iff \bar{h}_1 \triangleleft_{X'X} \bar{h}_2$, lead us to the notion of linearisability defined in Def. 8.

Theorem 35. $L_1 \sqsubseteq L_2$ implies $L_1 \preceq L_2$.

Proof. Consider C such that $C[L_1] \Downarrow$. We need to show $C[L_2] \Downarrow$. Because $C[L_1] \Downarrow$, Theorem 33 implies that there exists $h_1 \in \llbracket L_1 \rrbracket$ such that $\bar{h}_1 \in \llbracket C \rrbracket$. Because $L_1 \sqsubseteq L_2$, there exists $h_2 \in \llbracket L_2 \rrbracket$ with $h_1 \triangleleft_{PO}^* h_2$. Note that $\bar{h}_1 \triangleleft_{OP}^* \bar{h}_2$. By Lem. 34, $\bar{h}_2 \in \llbracket C \rrbracket$. Because $h_2 \in \llbracket L_2 \rrbracket$ and $\bar{h}_2 \in \llbracket C \rrbracket$, using Theorem 33 we can conclude $C[L_2] \Downarrow$. \square

Theorem 36. If $L_1 \sqsubseteq L_2$ then, for suitably typed L accessing disjoint part of the store than L_1 and L_2 , we have $L \cup L_1 \sqsubseteq L \cup L_2$.

6.3 Encapsulated linearisability

In this case libraries are being tested by clients that do not communicate with the parameter library explicitly.

Definition 37. [Encapsulated \sqsubseteq] Given libraries $L_1, L_2 : \Theta \rightarrow \Theta'$, we write $L_1 \sqsubseteq_{\text{enc}} L_2$ when, for all $L' : \emptyset \rightarrow \Theta$ and $\Theta' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$, if $\text{link } L' ; L_1$ in $(M_1 \parallel \dots \parallel M_N) \Downarrow$ then $\text{link } L' ; L_2$ in $(M_1 \parallel \dots \parallel M_N) \Downarrow$.

We shall call contexts of the above kind *encapsulated*, because the parameter library L' can no longer communicate directly with the client, unlike in Def. 23, where they shared methods in Θ'' . Consequently, $\llbracket \text{link } L' ; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket$ can be decomposed via parallel composition into two components, whose labels correspond to \mathcal{L} (parameter library) and \mathcal{K} (client) respectively.

Lemma 38 (Decomposition). *Suppose $L' : \emptyset \rightarrow \Theta$ and $\Theta' \vdash_K M_1 \parallel \dots \parallel M_N : \text{unit}$, where $\Theta \cap \Theta' = \emptyset$. Then, setting $C' \equiv \text{link } \emptyset ; - \text{ in } (M_1 \parallel \dots \parallel M_N)$, we have:*

$$\llbracket \text{link } L' ; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket = \{ h \in \mathcal{H}_{\Theta, \Theta'}^{\text{co}} \mid (h \upharpoonright \mathcal{L}) \in \llbracket L' \rrbracket, (h \upharpoonright \mathcal{K}) \in \llbracket C' \rrbracket \}.$$

Remark 39. Consider parameter library $L' : \emptyset \rightarrow \{m\}$ and client $\{m'\} \vdash_K M : \text{unit}$ with $m, m' \in \text{Meth}_{\text{unit} \rightarrow (\text{unit} \rightarrow \text{unit})}$, and suppose we insert in their context a “copycat” library L which implements m' as $m' = \lambda x.m.x$. Then the following scenario may seem to contradict encapsulation: $- M$ calls $m'()$; $- L$ calls $m()$; $- L'$ returns with $m(m'')$ to L ; $-$ and finally L copycats this return to M . However, by definition the latter copycat is done by L returning $m'(m''')$ to M , for some *fresh* name m''' , and recording internally that $m''' \mapsto \lambda x.m''x$. Hence, no methods of L' can leak to M and encapsulation holds.

Because of the above decomposition, the context semantics satisfies a stronger closure property than that already specified in Lem. 34, which in turn leads to the notion of encapsulated linearisability of Def. 12. The latter is defined in term of the symmetric reordering relation \diamond , which allows for swaps (in either direction) between moves from different threads if they are tagged with \mathcal{K} and \mathcal{L} respectively.

Moreover, we can show the following.

Lemma 40 (Encapsulated saturation). *Consider $X = \llbracket \text{link } L' ; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket$ (Definition 28). Then:*

- $-$ If $h \in X$ and $h (\triangleleft_{OP} \cup \diamond)^* h'$ then $h' \in X$.
- $-$ Let $s_1(t, x)_{OY} s_2(t, x')_{PY'} s_3 \in X$ be such that no move in s_2 comes from thread t . Then $Y = Y'$, i.e. inside a thread only O can switch between \mathcal{K} and \mathcal{L} .

Due to Theorem 33, the above property of contexts means that, in order to study termination in the encapsulated case, one can safely restrict attention to library traces satisfying a dual property to the one above, i.e. to elements of $\llbracket L \rrbracket_{\text{enc}}$. Note that $\llbracket L \rrbracket_{\text{enc}}$ can be obtained directly from our labelled transition system by restricting its single-threaded part to reflect the switching condition. Observe that Theorem 33 will still hold for $\llbracket L \rrbracket_{\text{enc}}$ (instead of $\llbracket L \rrbracket$), because we have preserved all the histories that are compatible with context histories. We are ready to prove correctness of encapsulated linearisability.

Theorem 41. $L_1 \sqsubseteq_{\text{enc}} L_2$ implies $L_1 \sqsim_{\text{enc}} L_2$.

Proof. Similarly to Theorem 35, except we invoke Lemma 40 instead of Lemma 34. \square

We next examine the behaviour of \sqsubseteq_{enc} with respect to library composition. In contrast to Section 6.2, we need to restrict composition for it to be compatible with encapsulation.

Remark 42. The general case of union does not conform with encapsulation in the sense that encapsulated testing of $L \cup L_i$ ($i = 1, 2$) according to Def. 37 may subject L_i to unencapsulated testing. For example, because method names of L and L_i are allowed to overlap, methods in L may call public methods from L_i as well as implementing abstract methods from L_i . This amounts to L playing the role of both \mathcal{K} and \mathcal{L} , which in addition can communicate with each other, as both are inside L .

Even if we make L and L_i non-interacting (i.e. without common abstract/public methods), if higher-order parameters are still involved, the encapsulated tests of $L \cup L_i$ can violate the encapsulation hypothesis for L_i . For instance, consider the methods $m_2, m'_1, m'_2 \in \text{Meths}_{\text{unit}, \text{unit}}$ and $m_1 \in \text{Meths}_{(\text{unit} \rightarrow \text{unit}), \text{unit}}$, and libraries $L_1, L_2 : \{m_1\} \rightarrow \{m_2\}$ and $L : \{m'_1\} \rightarrow \{m'_2\}$, as well as the unions $L \cup L_i : \{m_1, m_2\} \rightarrow \{m'_1, m'_2\}$. A possible trace in $\llbracket L \cup L_i \rrbracket_{\text{enc}}$ is this one:

$$\begin{aligned} h_i &= (1, \text{call } m_2())_{O\mathcal{K}} (1, \text{call } m_1(v))_{P\mathcal{L}} (1, \text{ret } m_1())_{O\mathcal{L}} \\ &\quad (1, \text{ret } m_2())_{P\mathcal{K}} (1, \text{call } m'_2())_{O\mathcal{K}} (1, \text{call } m'_1())_{P\mathcal{L}} (1, \text{call } v())_{O\mathcal{L}} \end{aligned}$$

which decomposes as $h_i = h' \mathrel{\mathbb{A}}_{\Pi, \emptyset}^{\sigma} h'_i$, with $\Pi = \{m_1, m_2, m'_1, m'_2\}$, $\sigma = 22221112$, $h' = (1, \text{call } m'_2())_{O\mathcal{K}} (1, \text{call } m'_1())_{P\mathcal{L}}$ and:

$$h'_i = (1, \text{call } m_2())_{O\mathcal{K}} (1, \text{call } m_1(v))_{P\mathcal{L}} (1, \text{ret } m_1())_{O\mathcal{L}} (1, \text{ret } m_2())_{P\mathcal{K}} (1, \text{call } v())_{O\mathcal{L}}$$

We now see that $h'_i \notin \llbracket L_i \rrbracket_{\text{enc}}$ as in the last move O is changing component from \mathcal{K} to \mathcal{L} .

We therefore look at compositionality for two specific cases: encapsulated sequencing (e.g. of $L : \Theta \rightarrow \Theta'$ with $L' : \Theta' \rightarrow \Theta''$) and disjoint union for first-order methods. Given $L : \Theta_1 \rightarrow \Theta_2$ and $L' : \Theta'_1 \rightarrow \Theta'_2$, we define their *disjoint union* $L \uplus L' = L \cup L' : (\Theta_1 \cup \Theta'_1) \rightarrow (\Theta_2 \cup \Theta'_2)$ under the assumption that $(\Theta_1 \cup \Theta_2) \cap (\Theta'_1 \cup \Theta'_2) = \emptyset$.

Theorem 43. Let $L_1, L_2 : \Theta_1 \rightarrow \Theta_2$ and $L : \Theta'_1 \rightarrow \Theta'_2$. If $L_1 \sqsubseteq_{\text{enc}} L_2$ then:

- assuming $\Theta'_2 = \Theta_1$, we have $L ; L_1 \sqsubseteq_{\text{enc}} L ; L_2$ and $L_1 ; L \sqsubseteq_{\text{enc}} L_2 ; L$;
- if $\Theta_1, \Theta_2, \Theta'_1, \Theta'_2$ are first-order then $L \uplus L_1 \sqsubseteq_{\text{enc}} L \uplus L_2$.

6.4 Relational linearisability

Finally, we examine relational linearisability (Def. 16). We begin with a suitable notion of relation \mathcal{R} . We next restrict encapsulated contextual testing to \mathcal{R} -closed contexts.

Definition 44. Let $\mathcal{R} \subseteq \mathcal{H}_{\emptyset, \Theta} \times \mathcal{H}_{\emptyset, \Theta}$ be a set closed under permutation of names in $\text{Meths} \setminus \Theta$. We say that $L : \emptyset \rightarrow \Theta$ is \mathcal{R} -closed if, for any h, h' such that $h \mathcal{R} h'$, if $h \in \llbracket L \rrbracket$ then $h' \in \llbracket L \rrbracket$.

Definition 45. [\mathcal{R} -closed encapsulated \sqsim] Given $L_1, L_2 : \Theta \rightarrow \Theta'$, we write $L_1 \sqsim_{\mathcal{R}} L_2$ if, for all \mathcal{R} -closed $L' : \emptyset \rightarrow \Theta$ and for all $\Theta' \vdash_{\mathcal{K}} M_1 \parallel \dots \parallel M_N : \text{unit}$, whenever link $L' ; L_1$ in $(M_1 \parallel \dots \parallel M_N) \Downarrow$ then we also have link $L' ; L_2$ in $(M_1 \parallel \dots \parallel M_N) \Downarrow$.

Theorem 46. $L_1 \sqsubseteq_{\mathcal{R}} L_2$ implies $L_1 \sqsubseteq_{\mathcal{R}} L_2$.

We conclude by showing that $\sqsubseteq_{\mathcal{R}}$ is also compositional in the sense proposed in [3]. Given $\mathcal{R}, \mathcal{G} \subseteq \mathcal{H} \times \mathcal{H}$, we say that L is $(\frac{\mathcal{R}}{\mathcal{G}})$ -closed if, for all $k \in \mathcal{H}$ and $h' \in \llbracket L \rrbracket_{\text{enc}}$, $(h' \upharpoonright \mathcal{K}) \mathcal{R} k$ implies that there is $h'' \in \llbracket L \rrbracket_{\text{enc}}$ with $(h'' \upharpoonright \mathcal{K}) = k$ and $(\overline{h'} \upharpoonright \mathcal{L}) \mathcal{G} (\overline{h''} \upharpoonright \mathcal{L})$.

Theorem 47. Let $\mathcal{R}, \mathcal{G} \subseteq \mathcal{H} \times \mathcal{H}$, $L_1, L_2 : \Theta_1 \rightarrow \Theta_2$ and $L : \Theta'_1 \rightarrow \Theta'_2$, such that $L_1 \sqsubseteq_{\mathcal{R}} L_2$. If L is suitably typed:

- if L is $(\frac{\mathcal{R}}{\mathcal{G}})$ -closed, we have $L ; L_1 \sqsubseteq_{\mathcal{G}} L ; L_2$; - $L_1 ; L \sqsubseteq_{\mathcal{R}} L_2 ; L$;
- if $\Theta_1, \Theta_2, \Theta'_1, \Theta'_2$ are first-order then $L \uplus L_1 \sqsubseteq_{\mathcal{R}^+} L \uplus L_2$, where $\mathcal{R}^+ = \{(s, s') \in \mathcal{H}_{\emptyset, \Theta_1 \cup \Theta'_1} \times \mathcal{H}_{\emptyset, \Theta_1 \cup \Theta'_1} \mid (s \upharpoonright \Theta_1) \mathcal{R} (s' \upharpoonright \Theta_1), (s \upharpoonright \Theta'_1) = (s' \upharpoonright \Theta'_1)\}$ and $s \upharpoonright \Theta$ is the largest subsequence of s belonging to $\mathcal{H}_{\emptyset, \Theta_1}$.

7 Related and future work

Since the work of Herlihy and Wing [12], linearisability has been consistently used as a correctness criterion for concurrent algorithms on a variety of data structures [18], and has yielded a variety of proof methods [5]. As mentioned in the Introduction, the field has focussed on libraries with methods of base-type inputs and outputs, with Cerone et al. recently catering for the presence of abstract as well as public methods [3]. An explicit connection between linearisability and refinement was made by Filipovic et al. in [6], where it was shown that, in base-type settings, linearisability and refinement coincide. Similar results have been proved in [4,9,17,3]. Our contributions herein are notions of linearisability that can serve as correctness criteria for libraries with methods of arbitrary higher-order types. Moreover, we relate them to refinement, thus establishing the soundness of linearisability, and demonstrate they are well-behaved with respect to library composition.

Verification of concurrent higher-order programs has been extensively studied outside of linearisability; we next mention works most closely related to linearisability reasoning. At the conceptual level, [6] proposed that the verification goal behind linearisability is observational refinement. In the same vein, [24] utilised logical relations as a direct method for proving refinement in a higher-order concurrent setting, while [23] introduced a program logic that builds on logical relations. On the other hand, proving conformance to a history specification has been addressed in [20] by supplying history-aware interpretations to off-the-shelf Hoare logics for concurrency. Other logic-based approaches for concurrent higher-order libraries, which do not use linearisability or any other notion of logical atomicity, include Higher-Order and Impredicative Concurrent Abstract Predicates [21,22].

One possible avenue for expansion of this work, following the example of [6], would be to identify language fragments where higher-order linearisability coincides with observational refinement. Based on the game semantic results of [7], such a correspondence may be possible to demonstrate already in the language examined herein.

The higher-order language we examined used memory in the form of references, which were global and moreover followed the standard memory model (sequential consistency). Therefore, future research also includes enriching the setting with dynamically allocated memory and expanding its reach to weak memory models. In the latter direction, our traces could need to be strengthened towards truly-concurrent structures, such as event-structures, following the recent examples of [2,14].

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A Big-step vs small-step reorderings

[3] defines linearisation in the general case using a “big-step” relation that applies a single permutation to the whole sequence. This contrasts with our definition as \triangleleft_{PO}^* , in which we combine multiple adjacent swaps. We show that the two definitions are equivalent.

Definition 48. [[3]] Let $h_1, h_2 \in \mathcal{H}_{\Theta, \Theta'}$ of equal length. We write $h_1 \triangleleft_{PO}^{\text{big}} h_2$ if there is a permutation $\pi : \{1, \dots, |h_1|\} \rightarrow \{1, \dots, |h_2|\}$ such that, writing $h_i(j)$ for the j -th element of h_i , for all j , we have $h_1(j) = h_2(\pi(j))$ and, for all $i < j$:

$$\begin{aligned} & ((\exists t. h_1(i) = (t, -) \wedge h_1(j) = (t, -)) \\ & \vee (\exists t_1, t_2. h_1(i) = (t_1, -)_P \wedge h_1(j) = (t_2, -)_O)) \implies h_2(i) < h_2(j) \end{aligned}$$

In other words, h_2 is obtained from h_1 by permuting moves in such a way that their order in threads is preserved and whenever a O -move occurred after an P -move in h_1 , the same must apply to their permuted copies in h_2 .

Lemma 49. $\triangleleft_{PO}^{\text{big}} = \triangleleft_{PO}^*$.

Proof. It is obvious that $\triangleleft_{PO}^* \subseteq \triangleleft_{PO}^{\text{big}}$, so it suffices to show the converse.

Suppose $h_1 \triangleleft_{PO}^{\text{big}} h_2$. Consider the set $X_{h_1, h_2} = \{h \mid h_1 \triangleleft_{PO}^* h, h \triangleleft_{PO}^{\text{big}} h_2\}$. Note that X_{h_1, h_2} is not empty, because $h_1 \in X_{h_1, h_2}$.

For two histories h', h'' , define $\delta(h', h'')$ to be the length of the longest common prefix of h' and h'' . Let $N = \max_h \{\delta(h, h_2) \mid h \in X_{h_1, h_2}\}$. Note that $N \leq |h_1| = |h_2|$.

- If $N = |h_2|$ then we are done, because $N = |h_2|$ implies $h_2 \in X_{h_1, h_2}$ and, thus, $h_1 \triangleleft_{PO}^* h_2$.
- Suppose $N < |h_2|$ and consider h such that $N = \delta(h, h_2)$. We are going to arrive at a contradiction by exhibiting $h' \in X_{h_1, h_2}$ such that $\delta(h', h_2) > N$. Because $N = \delta(h, h_2)$ and $N < |h_2|$, we have

$$\begin{aligned} h_2 &= a_1 \cdots a_N(t, m)u \\ h &= a_1 \cdots a_N(t_1, m_1) \cdots (t_k, m_k)(t, m)u', \end{aligned}$$

where $t_i \neq t$, because order in threads must be preserved. Consider

$$h' = a_1 \cdots a_N(t, m)(t_1, m_1) \cdots (t_k, m_k)u'.$$

Clearly $\delta(h', h_2) > N$ so, for a contradiction, it suffices to show that $h' \in X_{h_1, h_2}$. Note that because $h \triangleleft_{PO}^{\text{big}} h_2$, we must also have $h' \triangleleft_{PO}^{\text{big}} h_2$, because the new PO dependencies in h' (wrt h) caused by moving (t, m) forward are consistent with h_2 . Hence, we only need to show that $h \triangleleft_{PO}^* h'$. Let us distinguish two cases.

- If (t, m) is a P -move then, clearly, $h \triangleleft_{PO}^* h'$ (P -move moves forward).
- If (t, m) is an O -move then, because $h \triangleleft_{PO}^{\text{big}} h_2$, all of the (t_i, m_i) actions must be O -moves (otherwise their position wrt (t, m) would have to be preserved in h_2 and it isn't). Hence, $h \triangleleft_{PO}^* h'$, as required.

□

B Proofs from Section 6

Proof of Lemma 40. For the first claim, closure under \triangleleft_{OP} (resp. \diamond) follows from Lemma 34. (resp. Lemma 38).

Suppose $h = s_1(t, x)_{OY} s_2(t, x')_{PY'} s_3$ violates the second claim and $(t, x), (t, x')$ is the earliest such violation in h , i.e. no violations occur in s_1 . Observe that then h restricted to moves of the form $(t, z)_{XY'}$ would not be alternating, which contradicts the fact that $h \upharpoonright Y'$ is a history (Lemma 38). \square

Proof of Theorem 46. Consider C such that $C[L_1] \Downarrow$. We need to show $C[L_2] \Downarrow$. Since $C[L_1] \Downarrow$, by Theorem 33 there exists $h_1 \in \llbracket L_1 \rrbracket_{\text{enc}}$ such that $\overline{h_1} \in \llbracket C \rrbracket_{\text{enc}}$. Also, by Lemma 38, $(\overline{h_1} \upharpoonright \mathcal{K}) \in \llbracket C' \rrbracket_{\text{enc}}$ and $(\overline{h_1} \upharpoonright \mathcal{L}) \in \llbracket L' \rrbracket_{\text{enc}}$ for C', L' specified in that lemma. Because $L_1 \sqsubseteq_{\mathcal{R}} L_2$, there exists $h_2 \in \llbracket L_2 \rrbracket_{\text{enc}}$ such that $(h_1 \upharpoonright \mathcal{K}) \triangleleft_{PO}^* (h_2 \upharpoonright \mathcal{K})$ and $(\overline{h_1} \upharpoonright \mathcal{L}) \mathcal{R} (\overline{h_2} \upharpoonright \mathcal{L})$. Note that the former implies $(\overline{h_1} \upharpoonright \mathcal{K}) \triangleleft_{OP}^* (\overline{h_2} \upharpoonright \mathcal{K})$. Because $(\overline{h_1} \upharpoonright \mathcal{L}) \in \llbracket L' \rrbracket_{\text{enc}}$, $(\overline{h_1} \upharpoonright \mathcal{L}) \mathcal{R} (\overline{h_2} \upharpoonright \mathcal{L})$ and L' is \mathcal{R} -closed, we have $(\overline{h_2} \upharpoonright \mathcal{L}) \in \llbracket L' \rrbracket_{\text{enc}}$. On the other hand, because $(\overline{h_1} \upharpoonright \mathcal{K}) \in \llbracket C' \rrbracket_{\text{enc}}$ and $(\overline{h_1} \upharpoonright \mathcal{K}) \triangleleft_{OP}^* (\overline{h_2} \upharpoonright \mathcal{K})$ Lemma 40 implies $(\overline{h_2} \upharpoonright \mathcal{K}) \in \llbracket C' \rrbracket_{\text{enc}}$. Consequently, $(\overline{h_2} \upharpoonright \mathcal{K}) \in \llbracket C' \rrbracket_{\text{enc}}$ and $(\overline{h_2} \upharpoonright \mathcal{L}) \in \llbracket L' \rrbracket_{\text{enc}}$, so Lemma 38 entails $\overline{h_2} \in \llbracket C \rrbracket_{\text{enc}}$. Hence, since $h_2 \in \llbracket L_2 \rrbracket$ and $\overline{h_2} \in \llbracket C \rrbracket_{\text{enc}}$, we can conclude $C[L_2] \Downarrow$ by Theorem 33. \square

C Trace compositionality

In this section we demonstrate how the semantics of a library inside a context can be drawn by composing the semantics of the library and that of the context. The result played a crucial role in our arguments about linearisability and contextual refinement in Section 6.

Let us divide (reachable) evaluation stacks into two classes: L -stacks, which can be produced in the trace semantics of a library; and C -stacks, which appear in traces of a context.

$$\begin{aligned} \mathcal{E}_L &::= [] \mid m :: E :: \mathcal{E}'_L & \mathcal{E}_C &::= [] \mid m :: \mathcal{E}'_C \\ \mathcal{E}'_L &::= m :: \mathcal{E}_L & \mathcal{E}'_C &::= m :: E :: \mathcal{E}_C \end{aligned}$$

From the trace semantics definition we have that N -configurations in the semantics of a library feature evaluation stacks of the forms \mathcal{E}_L (in O -configurations) and \mathcal{E}'_L (in P -configurations): these we will call **L -stacks**. On the other hand, those produced from a context utilise **C -stacks** which are of the forms \mathcal{E}_C (in P -configurations) and \mathcal{E}'_C (in O -configurations).

From here on, when we write \mathcal{E} we will mean an L -stack or a C -stack. Moreover, we will call an N -configuration ρ an **L -configuration** (or a **C -configuration**), if $\rho = (\vec{\mathcal{C}}, \dots)$ and, for each i , $\mathcal{C}_i = (\mathcal{E}_i, \dots)$ with \mathcal{E}_i an L -stack (resp. a C -stack).

Let ρ, ρ' be N -configurations and suppose $\rho = (\vec{\mathcal{C}}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ is a C -configuration and $\rho' = (\vec{\mathcal{C}'}, \mathcal{R}', \mathcal{P}', \mathcal{A}', S')$ an L -configuration. We say that ρ and ρ' are **compatible**, written $\rho \approx \rho'$, if S and S' have disjoint domains and, for each i :

- $\mathcal{C}_i = (\mathcal{E}_C, M)$ and $\mathcal{C}'_i = (\mathcal{E}_L, -)$, or $\mathcal{C}_i = (\mathcal{E}'_C, -)$ and $\mathcal{C}'_i = (\mathcal{E}'_L, M)$.
- If the public and abstract names of \mathcal{C}_i are $(\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{K}})$ and $(\mathcal{A}_{\mathcal{L}}, \mathcal{A}_{\mathcal{K}})$ respectively, and those of \mathcal{C}'_i are $(\mathcal{P}'_{\mathcal{L}}, \mathcal{P}'_{\mathcal{K}})$ and $(\mathcal{A}'_{\mathcal{L}}, \mathcal{A}'_{\mathcal{K}})$, then $\mathcal{P}_{\mathcal{L}} = \mathcal{A}'_{\mathcal{L}}$, $\mathcal{P}_{\mathcal{K}} = \mathcal{A}'_{\mathcal{K}}$, $\mathcal{A}_{\mathcal{L}} = \mathcal{P}'_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{K}} = \mathcal{P}'_{\mathcal{K}}$.

- The private names of ρ (i.e. those in $\text{dom}(\mathcal{R}) \setminus \mathcal{P}_{\mathcal{L}} \setminus \mathcal{P}_{\mathcal{K}}$) do not appear in ρ' , and dually for the private names of ρ' .
- If $\mathcal{C}_i = (\mathcal{E}, \dots)$ and $\mathcal{C}'_i = (\mathcal{E}', \dots)$ then \mathcal{E} and \mathcal{E}' are in turn compatible, that is:
 - either $\mathcal{E} = m :: E :: \mathcal{E}_1$, $\mathcal{E}' = m :: \mathcal{E}'_1$ and $\mathcal{E}_1, \mathcal{E}'_1$ are compatible,
 - or $\mathcal{E} = m :: \mathcal{E}_1$, $\mathcal{E}' = m :: E :: \mathcal{E}'_1$ and $\mathcal{E}_1, \mathcal{E}'_1$ are compatible,
 - or $\mathcal{E} = \mathcal{E}' = []$.

Note, in particular, that if $\rho \asymp \rho'$ then ρ must be a context configuration, and ρ' a library configuration.

We next define a trace semantics on compositions of compatible such N -configurations. We use the symbol \oslash for configuration composition: we call this **external composition**, to distinguish it from the composition of ρ and ρ' we can obtain by merging their components, which we will examine later.

$$\begin{array}{c}
\frac{\rho_1 \Longrightarrow \rho'_1}{\rho_1 \oslash \rho_2 \longrightarrow \rho'_1 \oslash \rho_2} \text{ INT}_1 \quad \frac{\rho_2 \Longrightarrow \rho'_2}{\rho_1 \oslash \rho_2 \longrightarrow \rho_1 \oslash \rho'_2} \text{ INT}_2 \\
\frac{\rho_1 \xrightarrow{(t, \text{call } m(v))} \rho'_1 \quad \rho_2 \xrightarrow{(t, \text{call } m(v))} \rho'_2}{\rho_1 \oslash \rho_2 \longrightarrow \rho'_1 \oslash \rho'_2} \text{ CALL} \\
\frac{\rho_1 \xrightarrow{(t, \text{ret } m(v))} \rho'_1 \quad \rho_2 \xrightarrow{(t, \text{ret } m(v))} \rho'_2}{\rho_1 \oslash \rho_2 \longrightarrow \rho'_1 \oslash \rho'_2} \text{ RETN}
\end{array}$$

The INT rules above have side-conditions imposing that the resulting pairs of configurations are still compatible. Concretely, this means that the names created fresh in internal transitions do not match the names already present in the configurations of the other component. Note that external composition is not symmetric, due to the context/library distinction we mentioned.

Our next target is to show a correspondence between the above-defined semantic composition and the semantics obtained by (syntactically) merging compatible configurations. This will demonstrate that composing the semantics of two components is equivalent to first syntactically composing them and then evaluating the result. In order to obtain this correspondence, we need to make the semantics of syntactically composed configurations more verbose: in external composition methods belong either to the context or the library, and when e.g. the client wants to evaluate mm' , with m a library method, the call is made explicit and, more importantly, m' is replaced by a fresh method name. On the other hand, when we compose syntactically such a call will be done internally, and without refreshing m' .

To counter-balance the above mismatch, we extend the syntax of terms and evaluation contexts, and the operational semantics of closed terms as follows. The semantics will now involve quadruples of the form:

$$(E[M], \mathcal{R}_1, \mathcal{R}_2, S) \text{ written also } (E[M], \vec{\mathcal{R}}, S)$$

where the two repositories correspond to context and library methods respectively, so in particular $\text{dom}(\mathcal{R}_1) \cap \text{dom}(\mathcal{R}_2) = \emptyset$. Moreover, inside $E[M]$ we tag method names and lambda-abstractions with indices 1 and 2 to record which of the two components (context or library) is enclosing them: the tag 1 is used for the context, and 2 for the

library. Thus e.g. a name m^1 signals an occurrence of method m inside the context. Tagged methods are passed around and stored as ordinary methods, but their behaviour changes when they are applied. Moreover, we extend (tagged) evaluation contexts by explicitly marking return points of methods:

$$E ::= \bullet \mid \dots \mid \text{let } x = E \text{ in } M \mid mE \mid r := E \mid \langle m^i \rangle E$$

In particular, $E[M]$ may not necessarily be a (tagged) term, due to the return annotations. The new reduction rules are as follows (we omit indices when they are not used in the rules).

$$\begin{aligned} (E[i_1 \oplus i_2], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[i], \vec{\mathcal{R}}, S') \quad (i = i_1 \oplus i_2) \\ (E[t_{\text{id}}], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[t], \vec{\mathcal{R}}, S') \\ (E[\pi_j \langle v_1, v_2 \rangle], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[v_j], \vec{\mathcal{R}}, S') \\ (E[\text{if } i \text{ then } M_0 \text{ else } M_1], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[M_j], \vec{\mathcal{R}}, S) \quad (j = (i > 0)) \\ (E[\lambda^i x. M], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[m^i], \vec{\mathcal{R}} \uplus_i (m \mapsto \lambda x. M), S) \\ (E[m^i v], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[M\{v/x\}^i], \vec{\mathcal{R}}, S) \quad \text{if } \mathcal{R}_i(m) = \lambda x. M \\ (E[m^i v], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[\langle m^i \rangle M\{v'/x\}^{3-i}], \vec{\mathcal{R}}', S) \quad \text{if } \mathcal{R}_{3-i}(m) = \lambda x. M \text{ with} \\ &\quad \text{Meths}(v) = \{m_1, \dots, m_k\}, v' = v\{m'_j/m_j \mid 1 \leq j \leq k\}, \vec{\mathcal{R}}' = \vec{\mathcal{R}} \uplus_i \{m'_j \mapsto \lambda y. m_j y \mid 1 \leq j \leq k\} \\ (E[\langle m^i \rangle v], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[v^i], \vec{\mathcal{R}} \uplus_{3-i} \{m'_j \mapsto \lambda y. m_j y\}, S) \text{ with } m_j, m'_j \text{ and } v' \text{ as above} \\ (E[\text{let } x = v \text{ in } M], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[M\{v/x\}], \vec{\mathcal{R}}, S) \\ (E[!r], \vec{\mathcal{R}}, S) &\rightarrow'_t (E[S(r)], \vec{\mathcal{R}}, S) \\ (E[r := i], \vec{\mathcal{R}}, S) &\rightarrow'_t (E, \vec{\mathcal{R}}, S[r \mapsto i]) \\ (E[r := m^i], \vec{\mathcal{R}}, S) &\rightarrow'_t (E, \vec{\mathcal{R}}, S[r \mapsto m^i]) \end{aligned}$$

Above we write M^i for the term M with all its methods and lambdas tagged (or re-tagged) with i . Moreover, we use the convention e.g. $\vec{\mathcal{R}} \uplus_1 (m \mapsto \lambda x. M) = (\mathcal{R}_1 \uplus (m \mapsto \lambda x. M), \mathcal{R}_2)$. Note that the repositories need not contain tags as, whenever a method is looked up, we subsequently tag its body explicitly.

Thus, the computationally observable difference of the new semantics is in the rule for reducing $E[m^i v]$ when m is not in the domain of \mathcal{R}_i : this corresponds precisely to the case where e.g. a library method is called by the context with another method as argument. A similar behaviour is exposed when such a method is returning. However, this novelty merely adds fresh method names by η -expansions and does not affect the termination of the reduction.

Defining parallel reduction \Longrightarrow' in an analogous way to \Longrightarrow , we can show the following. We let a quadruple $(M_1 \parallel \dots \parallel M_N, \mathcal{R}, S)$ be *final* if $M_i = ()$ for all i , and we write $(M_1 \parallel \dots \parallel M_N, \mathcal{R}, S) \Downarrow$ if $(M_1 \parallel \dots \parallel M_N, \mathcal{R}, S)$ can reduce to some final quadruple; these notions are defined for $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, S)$ in the same manner.

Lemma 50. *For any legal $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, S)$, we have that $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, S) \Downarrow$ iff $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1 \cup \mathcal{R}_2, S) \Downarrow$.*

We now proceed to syntactic composition of N -configurations. Given a pair $\rho_1 \asymp \rho_2$, we define a single quadruple corresponding to their syntactic composition, called their **internal composition**, as follows. Let $\rho_1 = (\vec{C}, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1)$ and $\rho_2 = (\vec{C}', \mathcal{R}_2, \mathcal{P}_2, \mathcal{A}_2, S_2)$ and, for each i , $\mathcal{C}_i = (\mathcal{E}_i, X_i)$ and $\mathcal{C}'_i = (\mathcal{E}'_i, X'_i)$, with $\{X_i, X'_i\} = \{M_i, -\}$, and we let $k_i = 1$ just if $X_i = M_i$. We let the internal composition of ρ_1 and ρ_2 be the quadruple:

$$\rho_1 \bowtie \rho_2 = ((\mathcal{E}_1 \bowtie \mathcal{E}'_1)[M_1^{k_1}] \parallel \dots \parallel (\mathcal{E}_N \bowtie \mathcal{E}'_N)[M_N^{k_N}], \mathcal{R}_1, \mathcal{R}_2, S_1 \uplus S_2)$$

where compatible evaluation stacks $\mathcal{E}, \mathcal{E}'$ are composed into a single evaluation context $\mathcal{E} \bowtie \mathcal{E}'$, as follows.

$$\begin{aligned} (m :: E :: \mathcal{E}) \bowtie (m :: \mathcal{E}') &= (\mathcal{E} \bowtie \mathcal{E}')[E[\langle m \rangle \bullet]^1] \\ (m :: \mathcal{E}') \bowtie (m :: E :: \mathcal{E}) &= (\mathcal{E} \bowtie \mathcal{E}')[E[\langle m \rangle \bullet]^2] \end{aligned}$$

and $[] \bowtie [] = \bullet$. Unfolding the above, we have that, for example:

$$\begin{aligned} &[m_k, E_k, m_{k-1}, m_{k-2}, E_{k-2}, \dots, m_1, E_1] \\ &\bowtie [m_k, m_{k-1}, E_{k-1}, m_{k-2}, \dots, m_1] = E_1^1[\langle m_1^1 \rangle E_2^2[\dots E_k^{k'}[\langle m_k^{k'} \rangle \bullet] \dots]] \end{aligned}$$

where $k' = 2 - (k \bmod 2)$.

We proceed to fleshing out the correspondence. We observe that an L -configuration ρ can be the final configuration of a trace just if all its components are O -configurations with empty evaluation stacks. On the other hand, for C -configurations, we need to reach P -configurations with terms $()$. Thus, we call an N -configuration ρ *final* if $\rho = (\vec{C}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ and either $\mathcal{C}_i = ([], -)$ for all i , or $\mathcal{C}_i = ([], ())$ for all i .

Let us write $(\mathcal{S}_1, \hookrightarrow_1, \mathcal{F}_1)$ for the transition system induced from external composition, and $(\mathcal{S}_2, \hookrightarrow_2, \mathcal{F}_2)$ be the transition system derived from internal composition:

- $\mathcal{S}_1 = \{\rho \otimes \rho' \mid \rho \asymp \rho'\}$, $\mathcal{F}_1 = \{\rho \otimes \rho' \in \mathcal{S}_1 \mid \rho, \rho' \text{ final}\}$, and \hookrightarrow_1 the transition relation \longrightarrow defined previously.
- $\mathcal{S}_2 = \{(M_1 \parallel \dots \parallel M_N, \vec{\mathcal{R}}, S) \mid (M_1 \parallel \dots \parallel M_N, \mathcal{R}_1 \uplus \mathcal{R}_2, S) \text{ valid}\}$, $\mathcal{F}_2 = \{x \in \mathcal{S}_2 \mid x \text{ final}\}$, and \hookrightarrow_2 the transition relation \Longrightarrow' defined above.

A relation $R \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ is called a *bisimulation* if, for all $(x_1, x_2) \in R$:

- $x_1 \in \mathcal{F}_1$ iff $x_2 \in \mathcal{F}_2$,
- if $x_1 \hookrightarrow_1 x'_1$ then $x_2 \hookrightarrow_2 x'_2$ and $(x'_1, x'_2) \in R$,
- if $x_2 \hookrightarrow_2 x'_2$ then $x_1 \hookrightarrow_1 x'_1$ and $(x'_1, x'_2) \in R$.

Given $(x_1, x_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, we say that x_1 and x_2 are *bisimilar*, written $x_1 \sim x_2$, if $(x_1, x_2) \in R$ for some bisimulation R .

Lemma 51. *Let $\rho \asymp \rho'$ be compatible N -configurations. Then, $(\rho \otimes \rho') \sim (\rho \bowtie \rho')$.*

Recall we write \bar{h} for the O/P complement of the history h . We can now prove Theorem 33, which states that the behaviour of a library L inside a context C can be deduced by composing the semantics of L and C .

Theorem 33 Let $L : \Theta \rightarrow \Theta'$, $L' : 1 \rightarrow \Theta$, Θ_1 and Θ' , $\Theta_1 \vdash M_1, \dots, M_N : \text{unit}$, with L , L' and $M_1; \dots; M_N$ accessing pairwise disjoint parts of the store. Then, $\text{link } L'; L \text{ in } (M_1 \parallel \dots \parallel M_N) \Downarrow$ iff there is $h \in \llbracket L \rrbracket_N$ such that $\bar{h} \in \llbracket \text{link } L'; - \text{ in } (M_1 \parallel \dots \parallel M_N) \rrbracket$.

Proof. Let C be the context link $L'; - \text{ in } (M_1 \parallel \dots \parallel M_N)$, and suppose $(L) \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}_0, S_0)$ and $(L') \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}'_0, S'_0)$ with $\text{dom}(\mathcal{R}_0) \cap \text{dom}(\mathcal{R}'_0) = \text{dom}(S_0) \cap \text{dom}(S'_0) = \emptyset$. We set:

$$\begin{aligned}\rho_0 &= (([], -) \parallel \dots \parallel ([], -), \mathcal{R}_0, (\emptyset, \Theta'), (\Theta, \emptyset), S_0) \\ \rho'_0 &= (([], M_1) \parallel \dots \parallel ([], M_N), \mathcal{R}'_0, (\Theta, \emptyset), (\emptyset, \Theta'), S'_0)\end{aligned}$$

We pick these as the initial N -configurations for $\llbracket L \rrbracket_N$ and $\llbracket C \rrbracket$ respectively. Moreover, we have that $(L'; L) \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}''_0, S''_0)$ where $\mathcal{R}''_0 = \{(m, (\mathcal{R}_0 \uplus \mathcal{R}'_0)(m)\{!r/\bar{m}\}) \mid m \in \text{dom}(\mathcal{R}_0 \uplus \mathcal{R}'_0)\}$ and $S''_0 = (S_0 \uplus S'_0)\{!r/\bar{m}\} \uplus_s \{(r_i, m_i) \mid i = 1, \dots, n\}$, assuming $\Theta = \{m_1, \dots, m_n\}$ and r_1, \dots, r_n are fresh references of corresponding types. Hence, the initial triple for $\llbracket C[L] \rrbracket$ is taken to be $\phi_0 = (([], M_1) \parallel \dots \parallel ([], M_N), \mathcal{R}''_0, S''_0)$. On the other hand, $\rho'_0 \bowtie \rho_0 = (([], M_1) \parallel \dots \parallel ([], M_N), \mathcal{R}'_0, \mathcal{R}_0, S_0 \uplus S'_0)$ and, using also Lemma 50, we have that $\phi_0 \Downarrow$ iff $\rho'_0 \bowtie \rho_0 \Downarrow$.

Then, for the forward direction of the claim, from $\phi_0 \Downarrow$ we obtain that $\rho'_0 \bowtie \rho_0 \Downarrow$. From the previous lemma, we have that so does $\rho'_0 \otimes \rho_0$. From the latter reduction we obtain the required common history. Conversely, suppose $h \in \llbracket L \rrbracket_N$ and $\bar{h} \in \llbracket C \rrbracket$. WLOG, assume that $\text{Meths}(h) \cap (\text{dom}(\mathcal{R}_0) \cup \text{dom}(\mathcal{R}'_0)) \subseteq \Theta \cup \Theta_1 \cup \Theta'$ (we can appropriately alpha-covert \mathcal{R}_0 and \mathcal{R}'_0 for this). Then, ρ_0 and ρ'_0 both produce h , with opposite polarities. By definition of the external composite reduction, we then have that $\rho'_0 \otimes \rho_0$ reduces to some final state. By the previous lemma, we have that $\rho'_0 \bowtie \rho_0$ reduces to some final quadruple, which in turn implies that $\phi_0 \Downarrow$, i.e. $\text{link } L'; L \text{ in } (M_1 \parallel \dots \parallel M_N) \Downarrow$. \square

C.1 Lemma 50

We purpose to show that, for any legal $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, S)$, $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, S) \Downarrow$ iff $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1 \cup \mathcal{R}_2, S) \Downarrow$.

We prove something stronger. For any repository \mathcal{R} whose entries are of the form $(m, \lambda x.m'x)$, we define a directed graph $\mathcal{G}(\mathcal{R})$ where vertices are all methods appearing in \mathcal{R} , and (m, m') is a (directed) edge just if $\mathcal{R}(m) = \lambda x.m'x$. In such a case, we call \mathcal{R} an **expansion class** if $\mathcal{G}(\mathcal{R})$ is acyclic and all its vertices have at most one outgoing edge. Moreover, given an expansion class \mathcal{R} , we define the method-for-method substitution $\{\mathcal{R}\}$ that assigns to each vertex m of $\mathcal{G}(\mathcal{R})$ the (unique) leaf m' such that there is a directed path from m to m' in $\mathcal{G}(\mathcal{R})$. Let us write $\mathcal{L}(\mathcal{R})$ for the set of leaves of $\mathcal{G}(\mathcal{R})$. For any quadruple $\phi = (E_1[M_1] \parallel \dots \parallel E_N[M_N], \mathcal{R}_1, \mathcal{R}_2, S)$ and expansion class $\mathcal{R} \subseteq \mathcal{R}_1 \cup \mathcal{R}_2$, we define the triple:

$$\begin{aligned}\phi^{\# \mathcal{R}} &= (E_1[M_1] \parallel \dots \parallel E_N[M_N], \mathcal{R}_1 \cup \mathcal{R}_2, S)\{\mathcal{R}\} \\ &= (E_1[M_1]\{\mathcal{R}\} \parallel \dots \parallel E_N[M_N]\{\mathcal{R}\}, (\mathcal{R}_1 \cup \mathcal{R}_2)\{\mathcal{R}\}, S\{\mathcal{R}\})\end{aligned}$$

where $\mathcal{R}'\{\mathcal{R}\} = \{(m, \mathcal{R}'(m)\{\mathcal{R}\}) \mid m \in \text{dom}(\mathcal{R}' \setminus \mathcal{R}) \cup \mathcal{L}(\mathcal{R})\}$, $S\{\mathcal{R}\} = (S \upharpoonright \text{Refs}_{\text{int}}) \cup \{(r, S(r)\{\mathcal{R}\}) \mid r \in \text{dom}(S) \setminus \text{Refs}_{\text{int}}\}$, and $\underline{E[M]}$ is the term obtained from $E[M]$ by removing all tagging.

We next define a notion of indexed bisimulation between the transition systems produced from quadruples and triples respectively. Given an expansion class \mathcal{R} , a relation $R_{\mathcal{R}}$ between quadruples and triples is called an \mathcal{R} -bisimulation if, whenever $\phi_1 R_{\mathcal{R}} \phi_2$:

- ϕ_1 final implies ϕ_2 final
- ϕ_2 final implies $\phi_2 \Downarrow$
- $\phi_1 \Longrightarrow' \phi'_1$ implies $\phi_2 \Longrightarrow' \phi'_2$ and $\phi'_1 R_{\mathcal{R}'} \phi'_2$ for some expansion class $\mathcal{R}' \supseteq \mathcal{R}$
- $\phi_2 \Longrightarrow \phi'_2$ implies $\phi_1 \Longrightarrow'^* \phi'_1$ and $\phi'_1 R_{\mathcal{R}'} \phi'_2$ for some expansion class $\mathcal{R}' \supseteq \mathcal{R}$.

Thus, Lemma 50 directly follows from the next result.

Lemma 52. *For all expansion classes \mathcal{R} , the relation $R_{\mathcal{R}} =$*

$$\{(\phi, \phi^{\# \mathcal{R}}) \mid \phi = (E_1[M_1] \parallel \dots \parallel E_N[M_N], \vec{\mathcal{R}}, S) \text{ legal} \wedge \mathcal{R} \subseteq \mathcal{R}_1 \cup \mathcal{R}_2\}$$

is a bisimulation.

Proof. Suppose $\phi R_{\mathcal{R}} \phi^{\# \mathcal{R}}$. We note that finality conditions are satisfied: if ϕ is final then so is $\phi^{\# \mathcal{R}}$; while if $\phi^{\# \mathcal{R}}$ is final then all its contexts are from the grammar:

$$E' ::= \bullet \mid \langle m^i \rangle E'$$

so $\phi \Downarrow$ by acyclicity of $\mathcal{G}(\mathcal{R})$.

Suppose now $\phi \Longrightarrow \phi'$, say due to $(E_1[M_1], \mathcal{R}_1, \mathcal{R}_2, S) \rightarrow'_1 (E'_1[M'_1], \mathcal{R}'_1, \mathcal{R}'_2, S')$. In case the reduction is not a function call or return, then it can be clearly simulated by $\phi^{\# \mathcal{R}}$. Otherwise, suppose:

- $(E_1[m^i v], \vec{\mathcal{R}}, S) \rightarrow'_1 (E_1[M\{v/x\}^i], \vec{\mathcal{R}}, S)$. If $m \notin \text{dom}(\mathcal{R})$ then, writing \mathcal{R}_{12} for $\mathcal{R}_1 \cup \mathcal{R}_2$, the above can be simulated by $(\underline{E}_1[mv], \mathcal{R}_{12}, S)\{\mathcal{R}\} \rightarrow_1 (\underline{E}_1[M\{v/x\}], \mathcal{R}_{12}, S)\{\mathcal{R}\}$. If, on the other hand, $m \in \text{dom}(\mathcal{R})$, suppose $\mathcal{R}_i(m) = \lambda x. m'x$, then $M = m'x$ and $m\{\mathcal{R}\} = m'\{\mathcal{R}\}$ so we have:

$$\underline{E}_1[M\{v/x\}^i]\{\mathcal{R}\} = \underline{E}_1[(m'v)^i]\{\mathcal{R}\} = \underline{E}_1[(mv)^i]\{\mathcal{R}\}$$

- and $\underline{E}_1[(mv)^i] = \underline{E}_1[m^i v]$ by the way the semantics was defined, so $\phi'^{\# \mathcal{R}} = \phi^{\# \mathcal{R}}$.
- $(E_1[m^i v], \vec{\mathcal{R}}, S) \rightarrow'_1 (E_1[\langle m^i \rangle M\{v'/x\}^{3-i}], \vec{\mathcal{R}}', S)$, with $\mathcal{R}_{3-i}(m) = \lambda x. M$, $\text{Meths}(v) = \{m_1, \dots, m_k\}$, $v' = \{\tilde{m}'/\tilde{m}\}$ and $\vec{\mathcal{R}}' = \vec{\mathcal{R}} \uplus \{m'_j \mapsto \lambda x. m_j x \mid 1 \leq j \leq k\}$. Let $\mathcal{R}' = \mathcal{R} \uplus \{m'_j \mapsto \lambda x. m_j x \mid 1 \leq j \leq k\} \subseteq \mathcal{R}'_1 \cup \mathcal{R}'_2$. If $m \notin \text{dom}(\mathcal{R})$ then $(\underline{E}_1[mv], \mathcal{R}_{12}, S)\{\mathcal{R}\} \rightarrow_1 (\underline{E}_1[M\{v/x\}], \mathcal{R}_{12}, S)\{\mathcal{R}\}$, and we have:

$$\begin{aligned} \underline{E}_1[\langle m^i \rangle M\{v'/x\}^{3-i}]\{\mathcal{R}'\} &= \underline{E}_1[M\{v'/x\}]\{\mathcal{R}'\} \\ &= \underline{E}_1[M\{v/x\}]\{\mathcal{R}\} \end{aligned}$$

Moreover, $\mathcal{R}_{12}\{\mathcal{R}\} = (\mathcal{R}'_1 \cup \mathcal{R}'_2)\{\mathcal{R}'\}$ and $S\{\mathcal{R}\} = S\{\mathcal{R}'\}$, so $\phi'^{\# \mathcal{R}} = (\underline{E}_1[M\{v/x\}], \mathcal{R}_{12}, S)\{\mathcal{R}\}$. On the other hand, if $\mathcal{R}(m) = \lambda x. m''x$ then:

$$\begin{aligned} \underline{E}[\langle m^i \rangle M\{v'/x\}^{3-i}]\{\mathcal{R}'\} &= \underline{E}[m''v']\{\mathcal{R}'\} \\ &= \underline{E}[m''v]\{\mathcal{R}\} = \underline{E}[mv]\{\mathcal{R}\} \end{aligned}$$

so $\phi^{\# \mathcal{R}} = \phi'^{\# \mathcal{R}'}$.

– Finally, the cases for method-return reductions are treated similarly as above.

Suppose now $\phi^{\#R} \Longrightarrow \phi'$, where recall that we write ϕ as $(E_1[M_1] \parallel \dots \parallel E_N[M_N], \vec{R}, S)$. We show by induction on $\text{size}_R(E_1[M_1], \dots, E_N[M_N])$ that $\phi \Longrightarrow' \phi''$ and $\phi' R_{\mathcal{R}'} \phi''$ for some $\mathcal{R}' \supseteq \mathcal{R}$. The size-function we use measures the length of $\mathcal{G}(\mathcal{R})$ -paths that appear inside its arguments:

$$\begin{aligned} \text{size}_R(E_1[M_1], \dots, E_N[M_N]) &= \text{size}_R(E_1[M_1]) + \dots + \text{size}_R(E_N[M_N]) \\ \text{size}_R(E[M]) &= \sum_{m \in X_1} 2|m|_{\mathcal{R}} + \sum_{m \in X_2} 1 \end{aligned}$$

where X_1 is the multiset containing all occurrences of methods $m \in \text{dom}(\mathcal{R})$ inside $E[M]$ in call position (e.g. mM'), and X_2 contains all occurrences of methods $m \in \text{dom}(\mathcal{R})$ inside $E[M]$ in return position (i.e. $\langle m^i \rangle \dots$). We write $|m|_{\mathcal{R}}$ for the length of the unique directed path from m to a leaf in $\mathcal{G}(\mathcal{R})$. The fact that X_1, X_2 are multisets reflects that we count all occurrences of m in call/return positions. Suppose WLOG that the reduction to ϕ' is due to some $(\underline{E}_1[M_1], \mathcal{R}_{12}, S)\{\mathcal{R}\} \rightarrow_1 (E'[M'], \mathcal{R}', S')$. If the reduction happens inside $\underline{M}_1\{\mathcal{R}\}$ (this case also encompasses the base case of the induction) then the only case we need to examine is that of the reduction being a method call. In such a case, suppose we have $\underline{E}_1[M_1]\{\mathcal{R}\} = E[mv]$, $E' = E$, $M' = M\{v/x\}$ and $\mathcal{R}_{12}\{\mathcal{R}\}(m) = \lambda x.M$. Then, $\underline{E}_1[M_1] = \tilde{E}[\tilde{m}^i \tilde{v}]$ for some $\tilde{E}, \tilde{m}, \tilde{v}$ such that $\tilde{m}\{\mathcal{R}\} = m$, $\tilde{v}\{\mathcal{R}\} = v$ and $\tilde{E}\{\mathcal{R}\} = E$. If $m \neq \tilde{m}$ then, supposing $\mathcal{R}(\tilde{m}) = \lambda x.\tilde{m}'x$ we have the following cases:

- $(\tilde{E}[\tilde{m}^i \tilde{v}], \vec{R}, S) \rightarrow_1' (\tilde{E}[\tilde{m}^i \tilde{v}], \vec{R}, S) = \phi_1''$
- $(\tilde{E}[\tilde{m}^i \tilde{v}], \vec{R}, S) \rightarrow_1' (\tilde{E}[\langle \tilde{m}^i \rangle (\tilde{m}'v)^{3-i}], \vec{R}', S) = \phi_1''$, with $\vec{R}' = \vec{R} \uplus_{3-i} \{m'_j \mapsto \lambda x.m_jx \mid 1 \leq j \leq k\}$, etc.

Let ϕ'' be the extension of ϕ_1'' to an N -quadruple by using the remaining $E_i[M_i]$'s of ϕ , so that $\phi \Longrightarrow' \phi''$. In the first case above we have that $\phi''^{\#R} = \phi$, and in the latter that $\phi''^{\#R'} = \phi$ (with $\mathcal{R}' = \mathcal{R} \uplus \{m'_j \mapsto \lambda x.m_jx \mid 1 \leq j \leq k\}$), and we appeal to the IH.

Suppose now that $\tilde{m} = m$ and $\mathcal{R}_{12}(m) = \lambda x.\tilde{M}$. Then, one of the following is the case:

- $(\tilde{E}[\tilde{m}^i \tilde{v}], \vec{R}, S), \vec{R}, S) \rightarrow_1' (\tilde{E}[\tilde{M}\{v/x\}^i], \vec{R}, S) = \phi_1''$
- $(\tilde{E}[\tilde{m}^i \tilde{v}], \vec{R}, S) \rightarrow_1' (\tilde{E}[\langle \tilde{m}^i \rangle \tilde{M}\{v/x\}^{3-i}], \vec{R}', S) = \phi_1''$, with $\vec{R}' = \vec{R} \uplus_{3-i} \{m'_j \mapsto \lambda x.m_jx \mid 1 \leq j \leq k\}$, etc.

Extending ϕ_1'' to ϕ'' as above, in the former case we then have that $\phi''^{\#R} = \phi'$, and in the latter that $\phi''^{\#R'} = \phi'$, as required.

Finally, let us suppose that M_1 is some value v . Then, we can write E_1 as $E_1 = E_2[E']$, with E' coming from the grammar $E' ::= \bullet \mid \langle m^i \rangle E'$ and E_2 not being of the form $E''[\langle m^i \rangle \bullet]$. Observe that $\underline{E}_1 = \underline{E}_2$. If $E' = \bullet$ then by a case analysis on E_1 we can see that $\phi^{\#R}$ can simulate the reduction. Otherwise, $(E_2[E'[v]], \vec{R}, S) \rightarrow_1' (E_2[E''[v^i]], \vec{R}', S)$ whereby $E' = E''[\langle m^i \rangle \bullet]$ and $\vec{R}' = \vec{R} \uplus_{3-i} \{m'_j \mapsto \lambda x.m_jx \mid 1 \leq j \leq k\}$, etc. We have that

$$\phi_1'' = (\underline{E}_2[E''[v^i]], \vec{R}', S)\{\mathcal{R}'\} = (\underline{E}_2[E'[v]], \vec{R}, S)\{\mathcal{R}\}$$

and hence, extending ϕ_1'' to ϕ'' , we have $\phi''^{\#R'} = \phi^{\#R}$. We can now appeal to the IH. \square

C.2 Lemma 51

Let $\rho \prec \rho'$ be compatible N -configurations. Then, $(\rho \odot \rho') \sim (\rho \bowtie \rho')$.

We prove that the relation $R = \{(\rho_1 \odot \rho_2, \rho_1 \bowtie \rho_2) \mid \rho_1 \prec \rho_2\}$ is a bisimulation. Let us suppose that $(\rho_1 \odot \rho_2, \rho_1 \bowtie \rho_2) \in R$.

- Suppose $\rho_1 \odot \rho_2 \hookrightarrow_1 \rho'_1 \odot \rho'_2$. If the transition is due to (INT1) then $\rho_2 = \rho'_2$ and we can see that $\rho_1 \bowtie \rho_2 \Longrightarrow' \rho'_1 \bowtie \rho_2$. Similarly if the transition is due to (INT2). Suppose now we used instead (CALL), e.g. $\rho_1 \xrightarrow{(1, \text{call } m(v))} \rho'_1$ and $\rho_2 \xrightarrow{(1, \text{call } m(v))} \rho'_2$, and let us consider the case where $v \in \text{Meths}$ (the other case is simpler). Then, assuming $\rho_1 = (\mathcal{C}_1^1 \parallel \dots, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1)$ and $\rho_2 = (\mathcal{C}_1^2 \parallel \dots, \mathcal{R}_2, \mathcal{P}_2, \mathcal{A}_2, S_2)$, we have that either of the following scenarios holds, for some $x \in \{\mathcal{K}, \mathcal{L}\}$: $\mathcal{C}_1^1 = (\mathcal{E}_1, E[mm'])$, $\mathcal{C}_1^2 = (\mathcal{E}_2, -)$ and

$$\begin{aligned} & (\mathcal{E}_1, E[mm'], \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1) \xrightarrow{\text{call } m(v)}_1 \\ & (m :: E :: \mathcal{E}_1, \mathcal{R}_1 \uplus (v \mapsto \lambda x. m'x), \mathcal{P}_1 \cup_x \{v\}, \mathcal{A}_1, S_1) \\ & (\mathcal{E}_2, -, \mathcal{R}_2, \mathcal{P}_2, \mathcal{A}_2, S_2) \xrightarrow{\text{call } m(v)}_1 \\ & (m :: \mathcal{E}_2, M\{v/x\}, \mathcal{R}_2, \mathcal{P}_1, \mathcal{A}_1 \cup_x \{v\}, S_2) \end{aligned}$$

or its dual, where ρ_2 contains the code initiating the call. Focusing WLOG in the former case and setting $S = S_1 \uplus S_2$:

$$\begin{aligned} \rho_1 \bowtie \rho_2 &= ((\mathcal{E}_1 \bowtie \mathcal{E}_2)[E[m^1 m']] \parallel \dots, \mathcal{R}_1, \mathcal{R}_2, S) \\ &\hookrightarrow_2 ((\mathcal{E}_1 \bowtie \mathcal{E}_2)[E[\langle m^1 \rangle M\{v/x\}^2]] \parallel \dots, \mathcal{R}'_1, \mathcal{R}_2, S) \\ &= \rho'_1 \bowtie \rho'_2 \quad (\mathcal{R}'_1 = \mathcal{R}_1 \uplus (v \mapsto \lambda x. m'x)) \end{aligned}$$

The case for (RETN) is treated similarly.

- Suppose $\rho_1 \bowtie \rho_2 = (E[M_1] \parallel M_2 \parallel \dots \parallel M_N, \vec{\mathcal{R}}, S) \hookrightarrow_2 (E[M'_1] \parallel M_2 \parallel \dots \parallel M_N, \vec{\mathcal{R}}', S')$ and let $\rho_1 = ((\mathcal{E}_1, M''_1) \parallel \dots, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1)$ and $\rho_2 = ((\mathcal{E}_2, -) \parallel \dots, \mathcal{R}_2, \mathcal{P}_2, \mathcal{A}_2, S_2)$, where $(\mathcal{E}_1 \bowtie \mathcal{E}_2)[M''_1] = E[M_1]$. If the redex M_1 is not of the forms $M_1 = m^1 v$ or $M_1 = \langle m^1 \rangle v$, with $m \in \text{dom}(\mathcal{R}_2)$, then the reduction can clearly be simulated by $\rho_1 \odot \rho_2$ (internally, by ρ_1). Otherwise, similarly as above, the reduction can be simulated by a mutual call/return of m .

Finally, it is clear that $\rho_1 \odot \rho_2$ is final iff $\rho_1 \bowtie \rho_2$ is final. \square

D General compositionality

This compositionality result will allow us to compose histories of component libraries in order to obtain those of their composite library. Let $L_1 : \Theta_1 \rightarrow \Theta_2$ and $L_2 : \Theta'_1 \rightarrow \Theta'_2$. The semantic composition will be guided by two sets of names Π, \mathcal{P} . Π contains method names that are shared between by the respective libraries and their context. Thus $\Pi \supseteq \Theta_1 \cup \Theta'_1 \cup \Theta_2 \cup \Theta'_2$. The names in \mathcal{P} , on the other hand, will be used for private communication between L_1 and L_2 . Consequently, $\Pi \cap \mathcal{P}$ consists of names that can be used both for internal communication between L_1 and L_2 , and for contextual interactions, i.e. $\Pi \cap \mathcal{P} = (\Theta_1 \cup \Theta'_1) \cap (\Theta_2 \cup \Theta'_2)$.

Given $h_i \in \llbracket L_i \rrbracket (i = 1, 2)$, we define the *composition* of h_1 and h_2 , written $h_1 \mathbin{\mathbb{M}}_{II, P}^\sigma h_2$, as a partial operation depending on II, P and an additional parameter $\sigma \in \{0, 1, 2\}^*$ which we call a *scheduler*. It is given inductively as follows. We let $\epsilon \mathbin{\mathbb{M}}_{II, P}^\epsilon \epsilon = \epsilon$ and:

$$\begin{aligned} (t, \text{call } m(v)) s_1 \mathbin{\mathbb{M}}_{II, P}^{0\sigma} (t, \text{call } m(v)) s_2 &= s_1 \mathbin{\mathbb{M}}_{II, P'}^\sigma s_2 \\ (t, \text{ret } m(v)) s_1 \mathbin{\mathbb{M}}_{II, P}^{0\sigma} (t, \text{ret } m(v)) s_2 &= s_1 \mathbin{\mathbb{M}}_{II, P'}^\sigma s_2 \\ (t, \text{call } m(v))_{PY} s_1 \mathbin{\mathbb{M}}_{II, P}^{1\sigma} s_2 &= (t, \text{call } m(v))_{PY} (s_1 \mathbin{\mathbb{M}}_{II', P}^\sigma s_2) \\ (t, \text{ret } m(v))_{PY} s_1 \mathbin{\mathbb{M}}_{II, P}^{1\sigma} s_2 &= (t, \text{ret } m(v))_{PY} (s_1 \mathbin{\mathbb{M}}_{II', P}^\sigma s_2) \\ (t, \text{call } m(v))_{OY} s_1 \mathbin{\mathbb{M}}_{II, P}^{1\sigma} s_2 &= (t, \text{call } m(v))_{OY} (s_1 \mathbin{\mathbb{M}}_{II', P}^\sigma s_2) \\ (t, \text{ret } m(v))_{OY} s_1 \mathbin{\mathbb{M}}_{II, P}^{1\sigma} s_2 &= (t, \text{ret } m(v))_{OY} (s_1 \mathbin{\mathbb{M}}_{II', P}^\sigma s_2) \end{aligned}$$

along with the dual rules for the last four cases (i.e. where we schedule 2 in each case). Note that the definition uses sequences of moves that are suffixes of histories (such as s_i). The above equations are subject to the following side conditions:

- $\text{Meths}(v) \cap (II \cup P) = \emptyset$, $II' = II \uplus \text{Meths}(v)$ and $P' = P \uplus \text{Meths}(v)$;
- $m \in P$ in the 0-scheduling cases;
- $m \in II$ in the 1-scheduling cases and, also, $m \in II \setminus P$ in the third case (the P -call);
- in the 1-scheduling cases, we also require that the leftmost move with thread index t in s_2 is not a P -move.

History composition is a partial function: if the conditions above are not met, or h_1, h_2, σ are not of the appropriate form, then the composition is undefined. The above conditions ensure that the composed histories are indeed compatible and can be produced by composing actual libraries. For instance, the last condition corresponds to determinacy of threads: there can only be at most one component starting with a P -move in each thread t . We then have the following correspondence.

Theorem 53. *If $L_1 : \Theta_1 \rightarrow \Theta_2$ and $L_2 : \Theta'_1 \rightarrow \Theta'_2$ access disjoint parts of the store then*

$$\llbracket L_1 \cup L_2 \rrbracket_N = \{ h \in \mathcal{H} \mid \exists \sigma, h_1 \in \llbracket L_1 \rrbracket_N, h_2 \in \llbracket L_2 \rrbracket_N. h = h_1 \mathbin{\mathbb{M}}_{II_0, P_0}^\sigma h_2 \}$$

with $II_0 = \Theta_1 \cup \Theta_2 \cup \Theta'_1 \cup \Theta'_2$ and $P_0 = (\Theta_1 \cup \Theta'_1) \cap (\Theta_2 \cup \Theta'_2)$.

The rest of this section is devoted in proving the Theorem.

Recall that we examine library composition in the sense of union of libraries. This scenario is more general than the one of Appendix C as, during composition via union, the calls and returns of each of the component libraries may be caught by the other library or passed as a call/return to the outer context. Thus, the setting of this section comprises given libraries $L_1 : \Theta_1 \rightarrow \Theta_2$ and $L_2 : \Theta'_1 \rightarrow \Theta'_2$, such that $\Theta_2 \cap \Theta'_2 = \emptyset$, and relating their semantics to that of their union $L_1 \cup L_2 : (\Theta_1 \cup \Theta'_1) \setminus (\Theta_2 \cup \Theta'_2) \rightarrow \Theta_2 \cup \Theta'_2$.

Given configurations for L_1 and L_2 , in order to be able to reduce them together we need to determine which of their methods can be used for communication between them, and which for interacting with the external context, which represents player O in the game. We will therefore employ a set of method names, denoted by II and variants, to register those methods used for interaction with the external context. Another piece of

information we need to know is in which component in the composition was the last call played, or whether it was an internal call instead. This is important so that, when O (or P) has the choice to return to both components, in the same thread, we know which one was last to call and therefore has precedence. We use for this purpose sequences $w = (w_1, \dots, w_N)$ where, for each i , $w_i \in \{0, 1, 2\}^*$. Thus, if e.g. $w_1 = 2w'_1$, this would mean that, in thread 1, the last call to O , was done from the second component; if, on the other hand, $w_1 = 0w'_1$ then the last call in thread 1 was an internal one between the two components. Given such a w and some $j \in \{0, 1, 2\}$, for each index t , we write $j +_t w$ for $w[t \mapsto (jw_t)]$.

Let us fix libraries $L_1 : \Theta_1 \rightarrow \Theta_2$ and $L_2 : \Theta'_1 \rightarrow \Theta'_2$. Let ρ_1, ρ_2 be N -configurations, and in particular L -configurations, and suppose that $\rho_1 = (\vec{C}, \mathcal{R}, \mathcal{P}, \mathcal{A}, S)$ and $\rho_2 = (\vec{C}', \mathcal{R}', \mathcal{P}', \mathcal{A}', S')$. Moreover, let $\Theta_1 \cup \Theta_2 \cup \Theta'_1 \cup \Theta'_2 \subseteq \Pi$. We say that ρ_1 and ρ_2 are (w, Π) -**compatible**, written $\rho_1 \asymp_{\Pi}^w \rho_2$, if S, S' have disjoint domains and, for each i ,

- $\mathcal{C}_i = (\mathcal{E}'_L, M)$ and $\mathcal{C}'_i = (\mathcal{E}_L, -)$, or $\mathcal{C}_i = (\mathcal{E}_L, -)$ and $\mathcal{C}'_i = (\mathcal{E}'_L, M)$, or $\mathcal{C}_i = (\mathcal{E}_{L1}, -)$ and $\mathcal{C}'_i = (\mathcal{E}_{L2}, -)$.
- We have $\Theta_1 \subseteq \mathcal{A}_l, \Theta_2 \subseteq \mathcal{P}_{\mathcal{K}}, \Theta'_1 \subseteq \mathcal{A}'_{\mathcal{L}}, \Theta'_2 \subseteq \mathcal{P}'_{\mathcal{K}}$ and, setting

$$P = (\mathcal{P}_{\mathcal{K}} \uplus \mathcal{A}'_{\mathcal{L}}) \uplus (\mathcal{P}_{\mathcal{L}} \uplus \mathcal{A}'_{\mathcal{K}}) \uplus (\mathcal{P}'_{\mathcal{K}} \uplus \mathcal{A}_{\mathcal{L}}) \uplus (\mathcal{P}'_l \uplus \mathcal{A}_{\mathcal{K}})$$

we also have:

- $(\mathcal{P}_{\mathcal{L}} \uplus \mathcal{P}_{\mathcal{K}} \uplus \mathcal{A}_l \uplus \mathcal{A}_{\mathcal{K}}) \cap (\mathcal{P}'_{\mathcal{L}} \uplus \mathcal{P}'_{\mathcal{K}} \uplus \mathcal{A}'_l \uplus \mathcal{A}'_{\mathcal{K}}) = P \uplus (\Theta_1 \cap \Theta'_1)$,
 - $\Pi \cap P = (\Theta_2 \cup \Theta'_2) \cap (\Theta_1 \cup \Theta'_1)$,
 - $\Pi \cup P = \mathcal{P}_{\mathcal{L}} \cup \mathcal{P}_{\mathcal{K}} \cup \mathcal{P}'_l \cup \mathcal{P}'_{\mathcal{K}} \cup \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{K}} \cup \mathcal{A}'_{\mathcal{L}} \cup \mathcal{A}'_{\mathcal{K}}$.
 - The private names of \mathcal{R} do not appear in ρ_2 , and dually for the private names of \mathcal{R}' .
 - If $\mathcal{C}_i = (\mathcal{E}, \dots)$ and $\mathcal{C}'_i = (\mathcal{E}', \dots)$ then \mathcal{E} and \mathcal{E}' are w_i -compatible, that is, either $\mathcal{E} = \mathcal{E}' = []$ or:
 - $\mathcal{E} = m :: \mathcal{E}_1$ and $\mathcal{E}' \in \mathcal{E}_L$, with $m \in \Pi$, $w_i = 1u$ and $\mathcal{E}_1, \mathcal{E}'$ are u -compatible,
 - or $\mathcal{E} = m :: \mathcal{E}_1$ and $\mathcal{E}' = m :: E :: \mathcal{E}_2$, with $m \in P$, $w_i = 0u$ and $\mathcal{E}_1, \mathcal{E}_2$ are u -compatible,
 - or $\mathcal{E} = m :: E :: \mathcal{E}_1$ and $\mathcal{E}' \in \mathcal{E}_L$, with $m \in \Pi \setminus P$, $w_i = 1u$ and $\mathcal{E}_1, \mathcal{E}'$ are u -compatible,
- or the dual of one of the three conditions above holds.

Given $\rho_1 \asymp_{\Pi}^w \rho_2$, we let their external composition be denoted as $\rho_1 \otimes_{\Pi}^w \rho_2$ (and note that now the notation is symmetric for ρ_1 and ρ_2) and define the semantics for external

composition by these rules:

$$\begin{array}{c}
\frac{\rho_1 \Longrightarrow \rho'_1}{\rho_1 \otimes_{\Pi}^w \rho_2 \longrightarrow \rho'_1 \otimes_{\Pi}^w \rho_2} \text{INT}_1 \\
\frac{\rho_1 \xrightarrow{(t, \text{call } m(v))} \rho'_1 \quad \rho_2 \xrightarrow{(t, \text{call } m(v))} \rho'_2}{\rho_1 \otimes_{\Pi}^w \rho_2 \longrightarrow \rho'_1 \otimes_{\Pi}^{0+t} \rho'_2} \text{CALL } (m \in P) \\
\frac{\rho_1 \xrightarrow{(t, \text{ret } m(v))} \rho'_1 \quad \rho_2 \xrightarrow{(t, \text{ret } m(v))} \rho'_2}{\rho_1 \otimes_{\Pi}^{0+t} \rho_2 \longrightarrow \rho'_1 \otimes_{\Pi}^w \rho'_2} \text{RETN } (m \in P) \\
\frac{\rho_1 \xrightarrow{(t, \text{call } m(v))_{PY}} \rho'_1}{\rho_1 \otimes_{\Pi}^w \rho_2 \xrightarrow{(t, \text{call } m(v))_{PY}} \rho'_1 \otimes_{\Pi'}^{1+t} \rho_2} \text{PCALL}_1 \ (m \in \Pi \setminus P) \\
\frac{\rho_1 \xrightarrow{(t, \text{ret } m(v))_{PY}} \rho'_1}{\rho_1 \otimes_{\Pi}^{1+t} \rho_2 \xrightarrow{(t, \text{ret } m(v))_{PY}} \rho'_1 \otimes_{\Pi'}^w \rho_2} \text{PRETN}_1 \ (m \in \Pi) \\
\frac{\rho_1 \xrightarrow{(t, \text{call } m(v))_{OY}} \rho'_1}{\rho_1 \otimes_{\Pi}^w \rho_2 \xrightarrow{(t, \text{call } m(v))_{OY}} \rho'_1 \otimes_{\Pi'}^{1+t} \rho_2} \text{OCALL}_1 \ (m \in \Pi) \\
\frac{\rho_1 \xrightarrow{(t, \text{ret } m(v))_{OY}} \rho'_1}{\rho_1 \otimes_{\Pi}^{1+t} \rho_2 \xrightarrow{(t, \text{ret } m(v))_{OY}} \rho'_1 \otimes_{\Pi'}^w \rho_2} \text{ORETN}_1 \ (m \in \Pi \setminus P)
\end{array}$$

along with their dual counterparts (INT₂, XCALL₂, XRETN₂). The internal rules above have the same side-conditions on name privacy as before. Moreover, in (XRETN₂) and (XCALL₂), for X=O,P, we let $\Pi' = \Pi \uplus_t \text{Meths}(v)$ and impose that the t -th component of ρ_{3-i} be an O -configuration and $\text{Meths}(v) \cap \text{Meths}(\rho_{3-i}) = \emptyset$.

We can now show the following.

Lemma 54. *Let $\rho_1 \sim_{\Pi}^w \rho_2$ and suppose $\rho_1 \otimes_{\Pi}^w \rho_2 \xrightarrow{s}^* \rho'_1 \otimes_{\Pi'}^w \rho'_2$ for some sequence s of moves. Then, $\rho'_1 \sim_{\Pi'}^{w'} \rho'_2$.*

We next juxtapose the semantics of external composition to that obtained by internally composing the libraries and then deriving the multi-threaded semantics of the result. As before, we call the latter form *internal composition*. The traces we obtain are produced from a transition relation, written \Longrightarrow' , between configurations of the form $(\mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_N, \mathcal{R}_1, \mathcal{R}_2, \mathcal{P}, \mathcal{A}, S)$, also written $(\vec{\mathcal{C}}, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S)$. In particular, in each $\mathcal{C}_i = (\mathcal{E}_i, X_i)$ with $X_i = E_i[M_i]$ or $X_i = -$, E_i is selected from the extended evaluation contexts and \mathcal{E}_i is an *extended L-stack*, that is, of either of the following two forms:

$$\mathcal{E}_{\text{ext}} ::= [] \mid m^i :: E :: \mathcal{E}'_{\text{ext}} \quad \mathcal{E}'_{\text{ext}} ::= m :: \mathcal{E}_{\text{ext}}$$

where E is again from the extended evaluation contexts.

First, given u -compatible evaluation stacks $\mathcal{E}, \mathcal{E}'$, we construct a pair $\mathcal{E} \mathbin{\mathbb{M}}^u \mathcal{E}'$ consisting of an extended evaluation context and an extended L -stack, as follows. Given

$\mathcal{E} \mathrel{\mathbb{K}}^u \mathcal{E}' = (E', \mathcal{E}'')$:

$$\begin{aligned} (m :: E :: \mathcal{E}) \mathrel{\mathbb{K}}^{0u} (m :: \mathcal{E}') &= (E'[E[\langle m \rangle \bullet]^1], \mathcal{E}'') \\ (m :: \mathcal{E}) \mathrel{\mathbb{K}}^{0u} (m :: E :: \mathcal{E}') &= (E'[E[\langle m \rangle \bullet]^2], \mathcal{E}'') \\ (m :: \mathcal{E}) \mathrel{\mathbb{K}}^{1u} \mathcal{E}' &= \mathcal{E} \mathrel{\mathbb{K}}^{2u} (m :: \mathcal{E}') = (\bullet, m :: E' :: \mathcal{E}'') \\ (m :: E :: \mathcal{E}) \mathrel{\mathbb{K}}^{1u} \mathcal{E}' &= \mathcal{E} \mathrel{\mathbb{K}}^{2u} (m :: E :: \mathcal{E}') \\ &= (\bullet, m :: E'[E] :: \mathcal{E}'') \text{ if } \mathcal{E}' \in \mathcal{E}_L \end{aligned}$$

and $[] \mathrel{\mathbb{K}}^\epsilon [] = (\bullet, [])$.

For each pair $\rho_1 \asymp_{II}^w \rho_2$, we define a configuration corresponding to their syntactic composition as follows. Let $\rho_1 = (\mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_N, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1)$ and $\rho_2 = (\mathcal{C}'_1 \parallel \dots \parallel \mathcal{C}'_N, \mathcal{R}_2, \mathcal{P}_2, \mathcal{A}_2, S_2)$ and, for each i , $\mathcal{C}_i = (\mathcal{E}_i, X_i)$ and $\mathcal{C}'_i = (\mathcal{E}'_i, X'_i)$. If $\mathcal{E}_i \mathrel{\mathbb{K}}^u \mathcal{E}'_i = (E_i, \mathcal{E}''_i)$, we set:

$$\mathcal{C}_i \mathrel{\mathbb{K}}^u \mathcal{C}'_i = \begin{cases} (\mathcal{E}''_i, E_i[M^1]) & \text{if } X_i = M \text{ and } X'_i = - \\ (\mathcal{E}''_i, E_i[M^2]) & \text{if } X_i = - \text{ and } X'_i = M \\ (\mathcal{E}''_i, -) & \text{if } X_i = X'_i = - \end{cases}$$

We then let the internal composition of ρ_1 and ρ_2 be:

$$\rho_1 \mathrel{\mathbb{K}}_{II}^w \rho_2 = (\mathcal{C}_1 \mathrel{\mathbb{K}}^{w_1} \mathcal{C}'_1 \parallel \dots \parallel \mathcal{C}_N \mathrel{\mathbb{K}}^{w_N} \mathcal{C}'_N, \mathcal{R}_1, \mathcal{R}_2, \mathcal{P}', \mathcal{A}', S_1 \uplus S_2)$$

where we set $\mathcal{P}' = ((\mathcal{P}_{1\mathcal{L}} \uplus \mathcal{P}_{2\mathcal{L}}) \cap II, (\mathcal{P}_{1\mathcal{K}} \uplus \mathcal{P}_{2\mathcal{K}}) \cap II)$ and $\mathcal{A}' = ((\mathcal{A}_{1\mathcal{L}} \cup \mathcal{A}_{2\mathcal{L}}) \cap (II \setminus \mathcal{P}), (\mathcal{A}_{1\mathcal{K}} \uplus \mathcal{A}_{2\mathcal{K}}) \cap II)$.

Now, as expected, the definition of \Longrightarrow' builds upon \rightarrow'_t . The definition of the latter is given by the following rules.

$$\begin{aligned} &\frac{(E[M], \vec{\mathcal{R}}, S) \rightarrow'_t (E'[M'], \vec{\mathcal{R}}', S')}{(\mathcal{E}, E[M], \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) \rightarrow'_t (\mathcal{E}, E'[M'], \vec{\mathcal{R}}', \mathcal{P}, \mathcal{A}, S')} \text{ (INT')} \\ (\mathcal{E}, E[m^i v], \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) &\xrightarrow{\text{call } m(v')_{PY}}'_t (m^i :: E :: \mathcal{E}, -, \vec{\mathcal{R}}', \mathcal{P}', \mathcal{A}, S) \quad \text{(PQY')} \\ (m :: \mathcal{E}, v, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) &\xrightarrow{\text{ret } m(v')_{PY}}'_t (\mathcal{E}, -, \vec{\mathcal{R}}', \mathcal{P}', \mathcal{A}, S) \quad \text{(PAY')} \\ (\mathcal{E}, -, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) &\xrightarrow{\text{call } m(v)_{OY}}'_t (m :: \mathcal{E}, M\{v/x\}^i, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}', S) \quad \text{(OQY')} \\ (m^i :: E :: \mathcal{E}, -, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) &\xrightarrow{\text{ret } m(v)_{OY}}'_t (\mathcal{E}, E[v^i], \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}', S) \quad \text{(OAY')} \end{aligned}$$

The side-conditions are similar to those for the relation \rightarrow_t between ordinary configurations, with the following exceptions: in (PQY'), if $\text{Meths}(v) = \{m_1, \dots, m_k\}$ then $v' = v\{m'_j/m_j \mid 1 \leq j \leq k\}$, for fresh m'_j 's, and $\vec{\mathcal{R}}' = \vec{\mathcal{R}} \uplus_i \{m'_j \mapsto \lambda x. m_j x\}$; and in (PAY'), if $m \in \text{dom}(\mathcal{R}_i)$ then $\vec{\mathcal{R}}' = \vec{\mathcal{R}} \uplus_i \{m'_j \mapsto \lambda x. m_j x\}$, etc. Moreover, in (OQY') we have that $m \in \text{dom}(\mathcal{R}_i)$. Finally, we let

$$(\vec{\mathcal{C}}, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{(t,x)_{XY}}' (\vec{\mathcal{C}}[t \mapsto C'], \vec{\mathcal{R}}', \mathcal{P}', \mathcal{A}', S')$$

just if $(\mathcal{C}_t, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{x_{XY}}'_t (\mathcal{C}', \vec{\mathcal{R}}', \mathcal{P}', \mathcal{A}', S')$.

We next relate the transition systems induced by external (via \otimes) and internal composition (via \mathbb{K}). Let us write $(\mathcal{S}_1, \hookrightarrow_1, \mathcal{F}_1)$ for the transition system induced by external composition of compatible N -configurations (so \hookrightarrow_1 is \longrightarrow), and $(\mathcal{S}_2, \hookrightarrow_2, \mathcal{F}_2)$ be the one for internal composition (so \hookrightarrow_2 is \Longrightarrow). Finality of extended N -configurations $(\mathcal{C}_1 \parallel \dots \parallel \mathcal{C}_N, \vec{\mathcal{R}}, \dots)$ is defined as expected: all \mathcal{C}_i 's must be $([], -)$. A relation $R \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ is called a *bisimulation* if, for all $(x_1, x_2) \in R$:

- $x_1 \in \mathcal{F}_1$ iff $x_2 \in \mathcal{F}_2$,
- if $x_1 \hookrightarrow_1 x'_1$ then $x_2 \hookrightarrow_2 x'_2$ and $(x'_1, x'_2) \in R$,
- if $x_1 \xrightarrow{(t,x)_{XY}}_1 x'_1$ then $x_2 \xrightarrow{(t,x)_{XY}}_2 x'_2$ and $(x'_1, x'_2) \in R$,
- if $x_2 \hookrightarrow_2 x'_2$ then $x_1 \hookrightarrow_1 x'_1$ and $(x'_1, x'_2) \in R$,
- if $x_2 \xrightarrow{(t,x)_{XY}}_2 x'_2$ then $x_1 \xrightarrow{(t,x)_{XY}}_1 x'_1$ and $(x'_1, x'_2) \in R$.

Again, we say that x_1 and x_2 are *bisimilar*, and write $x_1 \sim x_2$, if there exists a bisimulation R such that $(x_1, x_2) \in R$.

Lemma 55. *Let $\rho \approx_H^w \rho'$ be compatible N -configurations. Then, $(\rho \otimes_H^w \rho') \sim (\rho \mathbb{K}_H^w \rho')$.*

Proof. We prove that the relation $R = \{(\rho_1 \otimes_H^w \rho_2, \rho_1 \mathbb{K}_H^w \rho_2) \mid \rho_1 \approx_H^w \rho_2\}$ is a bisimulation. Let us suppose that $(\rho_1 \otimes_H^w \rho_2, \rho_1 \mathbb{K}_H^w \rho_2) \in R$.

- Let $\rho_1 \otimes_H^w \rho_2 \xrightarrow{(t,x)} \rho'_1 \otimes_{H'}^{w'} \rho'_2$ with the transition being due to (XCALL₁), e.g. $\rho_1 \xrightarrow{(1, \text{call } m(v))} \rho'_1$ and $\rho'_2 = \rho_2, w' = 1 +_1 w$ and $H' = H \uplus_1 \text{Meths}(v)$, $\text{Meths}(v) = \{m'_1, \dots, m'_j\}$, and recall that $\text{Meths}(v) \cap \text{Meths}(\rho_2) = \emptyset$. Then, assuming $\rho_1 = (\mathcal{C}_1^1 \parallel \dots, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1)$, we have that one of the following holds, for some $\mathbf{x} \in \{\mathcal{K}, \mathcal{L}\}$:

$$\begin{aligned} \mathcal{C}_1^1 &= (\mathcal{E}_1, E[mv']) \text{ and } (\mathcal{C}_1^1, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1) \xrightarrow{\text{call } m(v)}_1 \\ &\quad (m :: E :: \mathcal{E}_1, -, \mathcal{R}_1 \uplus \{m'_j \mapsto \lambda x. m_j x \mid 1 \leq j \leq k\}, \mathcal{P}_1 \cup_{\mathbf{x}} \text{Meths}(v), \mathcal{A}_1, S_1) \\ \mathcal{C}_1^1 &= (\mathcal{E}_1, -) \text{ and } (\mathcal{C}_1^1, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1, S_1) \xrightarrow{\text{call } m(v)}_1 \\ &\quad (m^1 :: \mathcal{E}_1, mv, \mathcal{R}_1, \mathcal{P}_1, \mathcal{A}_1 \cup_{\mathbf{x}} \text{Meths}(v), S_1) \end{aligned}$$

In the former case, if $\rho_2 = ((\mathcal{E}_2, -) \parallel \dots, \mathcal{R}_2, \mathcal{P}_2, \mathcal{A}_2, S_2)$ with $\mathcal{E}_1 \mathbb{K}^{w_1} \mathcal{E}_2 = (E', \mathcal{E})$, we get:

$$\begin{aligned} \rho_1 \mathbb{K}_H^w \rho_2 &= ((\mathcal{E}, E'[E[mv']^1]) \parallel \dots, \mathcal{R}_1, \mathcal{R}_2, \mathcal{P}, \mathcal{A}, S) \\ &\xrightarrow{(1, \text{call } m(v))}_, \\ & (m^1 :: E'[E^1] :: \mathcal{E}, -) \parallel \dots, \mathcal{R}_1 \uplus \{m'_j \mapsto \lambda x. m_j x \mid 1 \leq j \leq k\}, \mathcal{R}_2, \mathcal{P}', \mathcal{A}, S) \end{aligned}$$

with \mathcal{P}, \mathcal{A} as in the definition of composition and $\mathcal{P}' = \mathcal{P} \cup_{\mathbf{x}} \text{Meths}(v)$, and the latter N -configuration equals $\rho'_1 \mathbb{K}_{H'}^{w'} \rho_2$. The other case is treated in the same manner, and we work similarly for (RET_N).

- On the other hand, if the transition is due to (CALL) or (RET_N) then we work as in the proof of Lemma 51.

- Suppose $\rho_1 \mathbb{A}_H^w \rho_2 = (\mathcal{C}_1 \parallel \dots, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{(1, \text{call } m(v))} (\mathcal{C}'_1 \parallel \dots, \vec{\mathcal{R}}', \mathcal{P}', \mathcal{A}', S)$. Then, assuming WLOG that $v \in \text{Meths}$, one of the following must be the case, for some $\mathbf{x} \in \{\mathcal{K}, \mathcal{L}\}$ and $i \in \{1, 2\}$:

$$\begin{aligned} \mathcal{C}_1 &= (\mathcal{E}, E[m^i v']) \text{ and } (\mathcal{C}_1, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) \xrightarrow{\text{call } m(v)} {}'_1 \\ &\quad (m^i :: E :: \mathcal{E}, \vec{\mathcal{R}} \uplus_i \{m'_j \mapsto \lambda x. m_j x \mid 1 \leq j \leq k\}, \mathcal{P} \cup_{\mathbf{x}} \text{Meths}(v), \mathcal{A}, S) \\ \mathcal{C}_1(\mathcal{E}, -) \text{ and } (\mathcal{C}_1, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S) &\xrightarrow{\text{call } m(v)} {}'_1 \\ &\quad (m :: \mathcal{E}, M\{v/x\}^i, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A} \cup_{\mathbf{x}} \text{Meths}(v), S) \end{aligned}$$

We only examine the former case, as the latter one is similar, and suppose that $i = 1$. Taking $\rho_j = (\mathcal{C}_1^j \parallel \dots, \mathcal{R}_j, \mathcal{P}_j, \mathcal{A}_j, S_i)$, for $j = 1, 2$, we have that $(\mathcal{C}_1^1, \mathcal{C}_1^2) = ((\mathcal{E}_1, E'[mv']), (\mathcal{E}_2, -))$, for some $E, \mathcal{E}_1, \mathcal{E}_2$ such that $\mathcal{E}_1 \mathbb{A}^{w_1} \mathcal{E}_2 = (E'', \mathcal{E})$ and $E = E''[E'^1]$. Moreover, taking $\mathcal{R}'_1 = \mathcal{R}_1 \uplus \{m'_j \mapsto \lambda x. m_j x \mid 1 \leq j \leq k\}$, $\mathcal{P}'_1 = \mathcal{P}_1 \uplus_{\mathbf{x}} \{v\}$, $w' = 1 +_1 w$ and $\Pi' = \Pi \uplus \text{Meths}(v)$ (note $\text{Meths}(v) = \{m'_1, \dots, m'_k\}$),

$$\rho_1 \otimes_H^w \rho_2 \xrightarrow{(1, \text{call } m(v))} ((m :: E' :: \mathcal{E}_1, -) \parallel \dots, \mathcal{R}'_1, \mathcal{P}'_1, \mathcal{A}_1, S_1) \otimes_{H'}^{w'} \rho_2 = \rho'_1 \otimes_{H'}^{w'} \rho_2$$

and $\rho'_1 \mathbb{A}_{H'}^{w'} \rho_2 = (\mathcal{C}'_1 \parallel \dots, \vec{\mathcal{R}}', \mathcal{P}', \mathcal{A}', S)$ as required. The case for return transitions is similar.

- On the other hand, if the transition out of $\rho_1 \mathbb{A}_H^w \rho_2$ does not have a label then we work as in the proof of Lemma 51.

Moreover, by definition of syntactic composition, $\rho_1 \otimes_H^w \rho_2$ is final iff $\rho_1 \mathbb{A}_H^w \rho_2$ is. \square

Given an N -configuration ρ and a history h , let us write $\rho \Downarrow h$ if $\rho \xrightarrow{h} \rho'$ for some final configuration ρ' . Similarly if ρ is of the form $(\vec{\mathcal{C}}, \vec{\mathcal{R}}, \mathcal{P}, \mathcal{A}, S)$. We have the following connections in history productions. The next lemma is proven in a similar fashion as Lemma 50.

Lemma 56. *For any legal $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, \mathcal{P}, \mathcal{A}, S)$ and history h , we have that $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1, \mathcal{R}_2, \mathcal{P}, \mathcal{A}, S) \Downarrow h$ iff $(M_1 \parallel \dots \parallel M_N, \mathcal{R}_1 \cup \mathcal{R}_2, \mathcal{P}, \mathcal{A}, S) \Downarrow h$.*

Lemma 57. *For any compatible N -configurations $\rho_1 \mathbb{A}_H^w \rho_2$ and history h , $(\rho_1 \otimes_H^w \rho_2) \Downarrow h$ iff:*

$$\exists h_1, h_2, \sigma. \rho_1 \Downarrow h_1 \wedge \rho_2 \Downarrow h_2 \wedge h = h_1 \mathbb{A}_{H, P}^\sigma h_2$$

where P is computed from ρ_1, ρ_2 and H as before.

Proof. We show that, for any compatible N -configurations $\rho_1 \mathbb{A}_H^w \rho_2$ and history suffix s , $(\rho_1 \otimes_H^w \rho_2) \Downarrow s$ iff:

$$\exists s_1, s_2, \sigma. \rho_1 \Downarrow s_1 \wedge \rho_2 \Downarrow s_2 \wedge s = s_1 \mathbb{A}_{H, P}^\sigma s_2$$

where P is computed from ρ_1, ρ_2 and H as in the beginning of this section.

The left-to-right direction follows from straightforward induction on the length of the reduction that produces s . For the right-to-left direction, we do induction on the

length of σ . If $\sigma = \epsilon$ then $s_1 = s_2 = s = \epsilon$. Otherwise, we do a case analysis on the first element of σ . We only look at the most interesting subcase, namely of $\sigma = 0\sigma'$. Then, for some $m \in P$:

$$s_1 = (t, \text{call } m(v))s'_1 \quad s_2 = (t, \text{call } m(v))s'_2$$

By $\rho_i \Downarrow s_i$ and $\rho_1 \approx_{II}^w \rho_2$ we have that $\rho_1 \otimes_{II}^w \rho_2 \longrightarrow \rho'_1 \otimes_{II}^{w'} \rho'_2$, where $w' = 0 +_t w$ and $\rho'_1 \approx_{II}^{w'} \rho'_2$. Also, $\rho'_i \Downarrow s'_i$ and $s = s'_1 \approx_{II, P'}^{\sigma'} s'_2$ so, by IH, $(\rho'_1 \otimes_{II}^{w'} \rho'_2) \Downarrow s$. \square

We can now prove the correspondence between the traces of component libraries and those of their union.

Theorem 53 Let $L_1 : \Theta_1 \rightarrow \Theta_2$ and $L_2 : \Theta'_1 \rightarrow \Theta'_2$ be libraries accessing disjoint parts of the store. Then,

$$\llbracket L_1 \cup L_2 \rrbracket_N = \{h \in \mathcal{H}^L \mid \exists \sigma, h_1 \in \llbracket L_1 \rrbracket_N, h_2 \in \llbracket L_2 \rrbracket_N. h = h_1 \approx_{II_0, P_0}^\sigma h_2\}$$

with $II_0 = \Theta_1 \cup \Theta_2 \cup \Theta'_1 \cup \Theta'_2$ and $P_0 = (\Theta_1 \cup \Theta'_1) \cap (\Theta_2 \cup \Theta'_2)$.

Proof. Let us suppose $(L_i) \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}_i, S_i)$, for $i = 1, 2$, with $\text{dom}(\mathcal{R}_1) \cap \text{dom}(\mathcal{R}_2) = \text{dom}(S_1) \cap \text{dom}(S_2) = \emptyset$. We set:

$$\begin{aligned} \rho_1 &= (([], -) \parallel \dots \parallel ([[], -), \mathcal{R}_1, (\emptyset, \Theta_2), (\Theta_1, \emptyset), S_1)) \\ \rho_2 &= (([], -) \parallel \dots \parallel ([[], -), \mathcal{R}_2, (\emptyset, \Theta'_2), (\Theta'_1, \emptyset), S_2)) \end{aligned}$$

We pick these as the initial configurations for $\llbracket L_1 \rrbracket_N$ and $\llbracket L_2 \rrbracket_N$ respectively. Then, $(L_1 \cup L_2) \xrightarrow{*}_{\text{lib}} (\epsilon, \mathcal{R}_0, S_0)$ where $\mathcal{R}_0 = \mathcal{R}_1 \uplus \mathcal{R}_2$ and $S_0 = S_1 \uplus S_2$, and we take

$$\rho_0 = ((([], -) \parallel \dots \parallel ([[], -), \mathcal{R}_0, (\emptyset, \Theta_2 \cup \Theta'_2), ((\Theta_1 \cup \Theta'_1) \setminus P_0, \emptyset), S_0))$$

as the initial N -configuration for $\llbracket L_1 \cup L_2 \rrbracket_N$. On the other hand, we have $\rho_1 \approx_{II_0}^\epsilon \rho_2 = ((([], -) \parallel \dots \parallel ([[], -), \mathcal{R}_1, \mathcal{R}_2, (\emptyset, \Theta_2 \cup \Theta'_2), ((\Theta_1 \cup \Theta'_1) \setminus P_0, S_0))$. From Lemma 56, we have that $\rho_0 \Downarrow h$ iff $\rho_1 \approx_{II_0}^\epsilon \rho_2 \Downarrow h$, for all h .

Pick a history h . For the forward direction of the claim, $\rho_0 \Downarrow h$ implies $\rho_1 \approx_{II_0}^\epsilon \rho_2 \Downarrow h$ which, from Lemma 55, implies $\rho_1 \otimes_{II_0}^\epsilon \rho_2 \Downarrow h$. We now use Lemma 57 to obtain h_1, h_2, σ such that $\rho_i \Downarrow h_i$ and $h = h_1 \approx_{II_0, P_0}^\sigma h_2$. Conversely, suppose that $h_i \in \llbracket L_i \rrbracket_N$ and $h = h_1 \approx_{II_0, P_0}^\sigma h_2$. WLOG assume that $(\text{Meths}(h_1) \cup \text{Meths}(h_2)) \cap (\text{dom}(\mathcal{R}_1) \cup \text{dom}(\mathcal{R}_2)) \subseteq II_0$ (or we appropriately alpha-covert \mathcal{R}_1 and \mathcal{R}_2). Then, $\rho_i \Downarrow h_i$, for $i = 1, 2$, and therefore $\rho_1 \otimes_{II_0}^\epsilon \rho_2 \Downarrow h$ by Lemma 57. By Lemma 55 we have that $\rho_1 \approx_{II_0}^\epsilon \rho_2 \Downarrow h$, which in turn implies that $\rho_0 \Downarrow h$, i.e. $h \in \llbracket L_1 \cup L_2 \rrbracket_N$. \square

E Composition congruence

E.1 Proof of Theorem 36

Proof. Assume $L_1 \sqsubseteq L_2$ and suppose $h_1 \in \llbracket L \cup L_1 \rrbracket$. By Theorem 53, $h_1 = h' \approx_{II, P}^\sigma h'_1$, where $h' \in \llbracket L \rrbracket$ and $h'_1 \in \llbracket L_1 \rrbracket$. Because $L_1 \sqsubseteq L_2$, there exists $h'_2 \in \llbracket L_2 \rrbracket$ such that $h'_1 \sqsubseteq h'_2$, i.e. $h'_1 \triangleleft_{PO}^* h'_2$. Note that some of the rearrangements necessary to transform h'_1 into h'_2 may concern actions shared by h'_1 and h' ; their polarity will then be different in h' . Let h'' be obtained by applying such rearrangements to h' . We claim that $h' \triangleleft_{OP}^* h''$.

Indeed, suppose that $(t', x')(t, x)_P$ are consecutive in h'_1 , but swapped in order to obtain h'_2 , and $(t, x)_P$ appears in h' as $(t, x)_O$. Now, the move (t', x') either appears in h_1 , or it appears in h' and gets hidden in h_1 . In every case, let s contain the moves of h' that are after (t', x') in the composition to h_1 , and before $(t, x)_O$. We have that $s(t, x)_O$ is a subsequence of h' and $h' \triangleleft_{OP}^* h''$ holds just if s contains no moves from t . But, if s contained moves from t then the rightmost one such would be some $(t, y)_P$. Moreover, in the composition towards h_1 , the move would be scheduled with 1. The latter would break the conditions for trace composition as, at that point, the corresponding subsequence of h'_1 has as leftmost move in t the P-move $(t, x)_P$. We can show similarly that $h' \triangleleft_{OP}^* h''$ holds in the case that the permutation in h'_1 is on consecutive moves $(t, x)_O(t', x')$. Finally, the rearrangements in h'_1 that do not affect moves shared with h' can be treated in a simpler way: e.g. in the case of $(t', x')(t, x)_P$ consecutive in h'_1 and swapped in h'_2 , if $(t, x)_P$ does not appear in h' then we can check that h' cannot contain any t -moves between (t', x') and (t, x) as the conditions for trace composition impose that only O is expected to play in that part of h' (and any t -move would swap this polarity). Now, since $h' \in \llbracket L \rrbracket$, Lemma 34 implies $h'' \in \llbracket L \rrbracket$. Take h_2 to be $h'' \bowtie_{II,P}^{\sigma'} h'_2$, where σ' is obtained from σ following these move rearrangements. We then have $h_2 \in \llbracket L \cup L_2 \rrbracket$. Moreover, $h_1 \sqsubseteq h_2$ thanks to $h'_1 \sqsubseteq h'_2$. Hence, $h_2 \in \llbracket L \cup L_2 \rrbracket$ and $h_1 \sqsubseteq h_2$. Thus, $L \cup L_1 \sqsubseteq L \cup L_2$. \square

E.2 Proof of Theorem 43

Proof. Let us consider the first sequencing case (the second one is dual), and assume that $L_1, L_2 : \Theta \rightarrow \Theta'$ and $L : \Theta'' \rightarrow \Theta$. Assume $L_1 \sqsubseteq_{\text{enc}} L_2$ and suppose $h_1 \in \llbracket L ; L_1 \rrbracket_{\text{enc}}$. By Theorem 53, $h_1 = h' \bowtie_{II,P}^{\sigma} h'_1$, where $h' \in \llbracket L \rrbracket$, $h'_1 \in \llbracket L_1 \rrbracket$ and method calls from Θ are always scheduled with 0. The fact that O cannot switch between \mathcal{L}/\mathcal{K} components in (threads of) h_1 implies that the same holds for h', h'_1 , hence $h' \in \llbracket L \rrbracket_{\text{enc}}$ and $h'_1 \in \llbracket L_1 \rrbracket_{\text{enc}}$. Because $L_1 \sqsubseteq_{\text{enc}} L_2$, there exists $h'_2 \in \llbracket L_2 \rrbracket_{\text{enc}}$ such that $h'_1 \sqsubseteq h'_2$, i.e. $h'_1 (\triangleleft_{PO} \cup \diamond)^* h'_2$. As before, some of the rearrangements necessary to transform h'_1 into h'_2 may concern actions shared by h'_1 and h' ; we need to check that these can lead to compatible $h'' \in \llbracket L \rrbracket_{\text{enc}}$. Let h'' be obtained by applying such rearrangements to h' . We claim that $h' \triangleleft_{OP}^* h''$. The transpositions covered by \triangleleft_{PO} are treated as in Lemma 36. Suppose now that $(t', x')_{PK}(t, x)_{OL}$ are consecutive in h'_1 but swapped in order to obtain h'_2 , and $(t, x)_{OL}$ appears in h' as $(t, x)_{PK}$. Now, the move (t', x') cannot appear in h' as it is in L_1 's \mathcal{K} -component (L is the \mathcal{L} -component of L_1). Let s contain the moves of h' that are after (t', x') in the composition to h_1 , and before $(t, x)_{PK}$. We claim that s contains no moves from t , so h' can be directly composed with h'_2 as far as this transposition is concerned. Indeed, if s contained moves from t then, taking into account the encapsulation conditions, the leftmost one such would be some $(t, y)_{OK}$. But the \mathcal{K} -component of L is L_1 , which contradicts the fact that the moves we consider are consecutive in h'_1 . Hence, taking h_2 to be $h'' \bowtie_{II,P}^{\sigma'} h'_2$, where σ' is obtained from σ following the \triangleleft_{PO} move rearrangements, we have $h_2 \in \llbracket L ; L_2 \rrbracket_{\text{enc}}$ and $h_1 \sqsubseteq_{\text{enc}} h_2$. Thus, $L ; L_1 \sqsubseteq_{\text{enc}} L ; L_2$.

The case of $L \uplus L_1 \sqsubseteq_{\text{enc}} L \uplus L_2$ is treated in a similar fashion. In this case, because of disjointness, the moves transposed in h'_1 do not have any counterparts in h' . Again, we consider consecutive moves $(t', x')_{PK}(t, x)_{OL}$ in h'_1 that are swapped in order to

obtain h'_2 . Let s contain the moves of h' that are after (t', x') in the composition to h_1 , and before (t, x) . As Θ_1, Θ'_1 is first-order, $(t, x)_{O\mathcal{L}}$ must be a return move and the t -move preceding it in h_1 must be the corresponding call. The latter is a move in h'_1 , which therefore implies that there can be no moves from t in s . Similarly for the other transposition case. \square

E.3 Proof of Theorem 47

Proof. For the first claim, suppose L is $(\mathcal{R})_{\mathcal{G}}$ -closed and $L_1 \sqsubseteq_{\mathcal{R}} L_2$. Consider $h_1 \in \llbracket L; L_1 \rrbracket_{\text{enc}}$. By Theorem 53, $h_1 = h' \mathbin{\text{\textcircled{\tiny R}}}^{\sigma}_{H,P} h'_1$, where $h' \in \llbracket L \rrbracket_{\text{enc}}$ and $h'_1 \in \llbracket L_1 \rrbracket_{\text{enc}}$. Since $L_1 \sqsubseteq_{\mathcal{R}} L_2$, there exists $h'_2 \in \llbracket L_2 \rrbracket_{\text{enc}}$ such that $(h'_1 \upharpoonright \mathcal{K}) \sqsubseteq (h'_2 \upharpoonright \mathcal{K})$ and $(\overline{h'_1} \upharpoonright \mathcal{L}) \mathcal{R} (\overline{h'_2} \upharpoonright \mathcal{L})$. By the permutation-closure of \mathcal{R} , we can pick h'_2 not to contain any common names with h' apart from those in the common moves of h'_1 and h' . Because L is $(\mathcal{R})_{\mathcal{G}}$ -closed, $h' \in \llbracket L \rrbracket_{\text{enc}}$ and $(h' \upharpoonright \mathcal{L}) = (\overline{h'_1} \upharpoonright \mathcal{L})$ and $(\overline{h'_1} \upharpoonright \mathcal{L}) \mathcal{R} (\overline{h'_2} \upharpoonright \mathcal{L})$, we can conclude that there exists $h'' \in \llbracket L \rrbracket_{\text{enc}}$ such that $(h'' \upharpoonright \mathcal{K}) = (\overline{h'_2} \upharpoonright \mathcal{L})$ and $(\overline{h'} \upharpoonright \mathcal{L}) \mathcal{G} (\overline{h''} \upharpoonright \mathcal{L})$. Applying the corresponding rearrangements to σ , we have that h'' and h'_2 are compatible, i.e. $(h'' \mathbin{\text{\textcircled{\tiny R}}}^{\sigma'}_{H,P} h'_2) \in \llbracket L; L_2 \rrbracket_{\text{enc}}$. Let $h_2 = h'' \mathbin{\text{\textcircled{\tiny R}}}^{\sigma'}_{H,P} h'_2$. We want to show $h_1 \sqsubseteq_{\mathcal{R}} h_2$. To that end, it suffices to make the following observations.

- We have $(h_1 \upharpoonright \mathcal{K}) \sqsubseteq (h_2 \upharpoonright \mathcal{K})$ because $(h_1 \upharpoonright \mathcal{K}) = (h'_1 \upharpoonright \mathcal{K})$, $(h'_1 \upharpoonright \mathcal{K}) \sqsubseteq (h'_2 \upharpoonright \mathcal{K})$ and $(h'_2 \upharpoonright \mathcal{K}) = (h_2 \upharpoonright \mathcal{K})$.
- We have $(\overline{h_1} \upharpoonright \mathcal{L}) \mathcal{G} (\overline{h_2} \upharpoonright \mathcal{L})$ because $(\overline{h_1} \upharpoonright \mathcal{L}) = (\overline{h'} \upharpoonright \mathcal{L})$, $(\overline{h'} \upharpoonright \mathcal{L}) \mathcal{G} (\overline{h''} \upharpoonright \mathcal{L})$ and $(\overline{h''} \upharpoonright \mathcal{L}) = (\overline{h_2} \upharpoonright \mathcal{L})$.

Consequently $L; L_1 \sqsubseteq_{\mathcal{G}} L; L_2$.

Suppose now $L_1 \sqsubseteq_{\mathcal{R}} L_2$. Consider $h_1 \in \llbracket L_1; L \rrbracket_{\text{enc}}$, i.e. $h_1 = h'_1 \mathbin{\text{\textcircled{\tiny R}}}^{\sigma}_{H,P} h'$, where $h'_1 \in \llbracket L_1 \rrbracket_{\text{enc}}$ and $h' \in \llbracket L \rrbracket_{\text{enc}}$. Because $L_1 \sqsubseteq_{\mathcal{R}} L_2$, there exists $h'_2 \in \llbracket L_2 \rrbracket_{\text{enc}}$ such that $h'_1 \sqsubseteq_{\mathcal{R}} h'_2$, i.e. $(h'_1 \upharpoonright \mathcal{K}) \sqsubseteq (h'_2 \upharpoonright \mathcal{K})$ and $(\overline{h'_1} \upharpoonright \mathcal{L}) \mathcal{R} (\overline{h'_2} \upharpoonright \mathcal{L})$. Define h'' to be h' in which $(\overline{h'} \upharpoonright \mathcal{L}) = (\overline{h'_1} \upharpoonright \mathcal{K})$ was modified by applying the same rearrangements as those witnessing $(h'_1 \upharpoonright \mathcal{K}) \sqsubseteq (h'_2 \upharpoonright \mathcal{K})$. Consequently $h' \triangleleft_{PO}^* h''$. By Lemma 40, $h'' \in \llbracket L \rrbracket_{\text{enc}}$. Moreover, $(\overline{h''} \upharpoonright \mathcal{L}) = (\overline{h'_2} \upharpoonright \mathcal{L})$. Consequently, h'_2 and h'' are compatible for the corresponding σ' . Let $h_2 = h'_2 \mathbin{\text{\textcircled{\tiny R}}}^{\sigma'}_{H,P} h'' \in \llbracket L_2; L \rrbracket$. Then we get:

- $(h_1 \upharpoonright \mathcal{K}) = (h'_1 \upharpoonright \mathcal{K}) = (\overline{h''} \upharpoonright \mathcal{K}) = (h_2 \upharpoonright \mathcal{K})$;
- $(\overline{h_1} \upharpoonright \mathcal{L}) = (\overline{h'_1} \upharpoonright \mathcal{L})$, $(\overline{h'_1} \upharpoonright \mathcal{L}) \mathcal{R} (\overline{h'_2} \upharpoonright \mathcal{L})$, $(\overline{h'_2} \upharpoonright \mathcal{L}) = (\overline{h_2} \upharpoonright \mathcal{L})$.

Consequently $h_1 \sqsubseteq_{\mathcal{R}} h_2$ and, hence, $L_1; L \sqsubseteq_{\mathcal{R}} L_2; L$.

For the last claim, we observe that because of the type-restrictions, the elements of $\llbracket L \uplus L_i \rrbracket_{\text{enc}}$ are interleavings of histories from $\llbracket L \rrbracket_{\text{enc}}$ and $\llbracket L_i \rrbracket_{\text{enc}}$. Consider now $h_1 \in \llbracket L \uplus L_1 \rrbracket_{\text{enc}}$, i.e. $h_1 = h' \mathbin{\text{\textcircled{\tiny R}}}^{\sigma}_{H,P} h'_1$, where $h'_1 \in \llbracket L_1 \rrbracket_{\text{enc}}$ and $h' \in \llbracket L \rrbracket_{\text{enc}}$, and let $h'_2 \in \llbracket L_2 \rrbracket_{\text{enc}}$ be such that $h'_1 \sqsubseteq_{\mathcal{R}} h'_2$. From our previous observation, we have that h' can still be composed with h'_2 , for appropriate σ' . Thus, taking $h_2 = h' \mathbin{\text{\textcircled{\tiny R}}}^{\sigma'}_{H,P} h'_2$, we have $h_2 \in \llbracket L \uplus L_2 \rrbracket_{\text{enc}}$ and, moreover, $h'_1 \sqsubseteq_{\mathcal{R}} h'_2$ implies $h_1 \sqsubseteq_{\mathcal{R}^+} h_2$. \square